NST 1

Tonbridge School, 24 May 2015

1. Let n points be given inside a rectangle R such that no two of them line on a line parallel to the sides of R. The rectangle is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

2. Define the function $f\colon (0,1) \longrightarrow (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} \text{ if } x < \frac{1}{2} \\ \\ x^2 \text{ if } x \geq \frac{1}{2}. \end{cases}$$

   Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences (a_n) and (b_n) by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there is a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

3. Consider a circle Γ with three fixed points A, B and C on Γ. Also fix a real number $\lambda \in (0,1)$. Let P denote a variable point on Γ with $P \notin \{A, B, C\}$, and let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that, as P varies, the point Q lies on a fixed circle.