

NST3 2009

Eton May 21, Oundle May 26

1. Let ABC be an acute-angled triangle, and M be a point in its plane distinct from the vertices. Show that the vector equation

$$\frac{a}{MA}\mathbf{MA} + \frac{b}{MB}\mathbf{MB} + \frac{c}{MC}\mathbf{MC} = \mathbf{0}$$

holds if, and only if, M is the orthocentre of ABC .

2. Let

$$f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$$

be a polynomial with integral coefficients of degree at least 3. Suppose that a_0 is even, and that $a_i + a_{n-i}$ is even for all i with $1 \leq i \leq n-1$. Suppose that $f = gh$ where g and h are integral polynomials, $\deg g \leq \deg h$, and all coefficients of h are odd. Show that f has an integral root.

3. Three travel companies operate a transport network between n cities. The transport between each pair of cities is a monopoly of a single travel company. Prove that, if $n \geq 11$, there must be a round trip through four cities with each journey organized by the same company, but that if $n \leq 10$, this is not necessarily the case.