NST 3

1 June 2010

1. Let $n \geq 2$ be an integer. At every point of the co-ordinate plane with integral co-ordinates (i, j) we write $i + j$ modulo n (an integer in the range $[0, n-1]$). Find all pairs (a, b) of positive integers such that the rectangle with vertices $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$ has both the following properties.

   (a) The remainders $0, 1, \ldots, n-1$ are each written the same number of times in its interior.

   (b) The remainders $0, 1, \ldots, n-1$ are each written the same number of times on its boundary.

2. Find all real numbers t for which there exist real numbers x, y, z such that each of the following equations is satisfied.

$$3x^2 + 3xz + z^2 = 1, \quad 3y^2 + 3yz + z^2 = 4, \quad x^2 - xy + y^2 = t.$$

3. Let p be a prime number which leaves remainder 3 on division by 4. Consider the equation

$$(p+2)x^2 - (p+1)y^2 + px + (p+2)y = 1.$$

   (a) Suppose that x, y are positive integers which satisfy the equation. Show that p divides x.

   (b) Show that the equation has infinitely many solutions in positive integers.

Each problem is worth 7 points.
Time: 4 hours 30 minutes.