Next Selection Test: Paper 3

Oundle School, Northamptonshire

5^{th} June 2012

1. Graphistan has 2011 cities; the company Graph Air sells flights in one direction only between every pair of cities. Let k be the maximum, taken over all cities in Graphistan, of the difference (in absolute value) between the number of Graph Air flights available from that city, and the number available into that city. Find the largest value of k for which it is guaranteed to be possible to travel from any city in Graphistan to any other using only Graph Air's services.

2. Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself, such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

3. Let n be an integer number greater than 2, and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that

$$\sum_{i=1}^{n} \frac{1}{x_i + 1} = 1.$$

If α is a real number greater than 1, show that

$$\sum_{i=1}^{n} \frac{1}{x_i^{\alpha} + 1} \geq \frac{n}{(n-1)^{\alpha} + 1}.$$

When does equality hold?

Each question is worth seven marks.
Time permitted: 4 hours, 30 minutes.