NST 3

Oundle, May 27 2014

1. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is also the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

2. Let n be a positive integer, and let A be a subset of $\{1, 2, \ldots, n\}$. An A-*partition* of n into k *parts* is a representation of n as sum $n = a_1 + a_2 + \cdots + a_k$, where the parts $a_1, a_2, \ldots, a_k$ belong to A and are not necessarily distinct. The number of *different parts* in such a partition is the number of (distinct) elements of the set $\{a_1, a_2, \ldots, a_k\}$.

   We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

3. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

   $$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) \equiv 2(x^3 - mx + 1)P(x).$$

   The notation $\equiv$ *denotes equality of polynomials.*