NST 3

Tonbridge School, 26 May 2015

1. The acute triangle ABC with $AB \neq AC$ has circumcircle Γ, circumcenter O, and orthocenter H. The midpoint of BC is M, and the extension of the median AM intersects Γ at N. The circle of diameter AM intersects Γ again at A and P. Show that the lines AP, BC, and OH are concurrent if and only if $AH = HN$.

2. Given a positive real number t, determine the sets A of real numbers containing t, for which there exists a set B (depending on A) with $|B| \geq 4$ such that $AB = \{ab \mid a \in A, b \in B\}$ is a finite arithmetic progression.

3. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geq n + 2$. On the segment $I = [0, \prod_i a_i]$ of the real line, mark all integers which are divisible by at least one of the numbers $a_1, a_2, \ldots, a_n$. These points break I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.