# NST 4

2 June 2010

1. Consider the sequence $(a_i)$ such that $a_0 = 4$, $a_1 = 22$ and
$$a_n - 6a_{n-1} + a_{n-2} = 0$$
for $n \geq 2$. Prove that there are integral sequences $(x_i)$, $(y_i)$ such that
$$a_n = \frac{y_n^2 + 7}{x_n - y_n}$$
for every $n \geq 0$.

2. The triangle $ABC$ is not isosceles. Let the inscribed circle $\Gamma$ have centre $I$ and touch the sides at $A_1, B_1$ and $C_1$ in the natural notation. Let $AA_1$ meet $\Gamma$ again at $A_2$, and define $B_2$ in similar fashion. The points $A_3$ on $B_1C_1$ and $B_3$ on $A_1C_1$ are such that $A_1A_3$ and $B_1B_3$ are angle bisectors in triangle $A_1B_1C_1$. Prove the following statements.
   (a) $A_2A_3$ bisects $\angle B_1A_2C_1$.
   (b) Let $P$ and $Q$ be the intersection points of the circumcircles of triangles $A_1A_2A_3$ and $B_1B_2B_3$, then $I$ lies on the line $PQ$.

3. The list $a_1, a_2, \ldots, a_n$ is a permutation of $1, 2, \ldots, n$. A *move* is a rearrangement of a permutation where two consectutive runs are exchanged. To be explicit one could apply a move replacing
$$a_1, \ldots, a_i, \underbrace{a_{i+1}, \ldots a_{i+p}}_{A}, \underbrace{a_{i+p+1}, \ldots a_{i+q}}_{B}, a_{i+q+1}, \ldots, a_n$$
by
$$a_1, \ldots, a_i, \underbrace{a_{i+p+1}, \ldots a_{i+q}}_{B}, \underbrace{a_{i+1}, \ldots a_{i+p}}_{A}, a_{i+q+1}, \ldots, a_n.$$
Find the least number of moves necessary to reorder $n, n-1, \ldots, 1$ into $1, 2, \ldots, n$.

*Each problem is worth 7 points.*
*Time: 4 hours 30 minutes.*