NST 4

Oundle, 29 May 2013

1. A *polyomino* is a finite union of square unit cells in a chequered plane, each two cells c and c' of which are connected by a finite chain of cells in the union,
$$c = c_0, c_1, \ldots, c_k = c'$$
where c_{i-1} and c_i are different but share a common edge for $i = 1$ to k. Prove that a polyomino P satisfying the following two conditions is a square:
   (a) At least one non-empty rectangle can be tiled with copies of P.
   (b) For each rectangle which can be tiled with copies of P, the tiling is unique up to reflections which preserve the rectangle.

2. Let x be an irrational number. We say that a positive integer n is x-admissible if there exists a positive integer $k \leq n$ such that $\{kx - \frac{1}{2}\} < \frac{1}{\sqrt{n}}$. Are there irrational numbers x such that the set of non-x-admissible positive integers is infinite? *Here* $\{y\} = y - \lfloor y \rfloor$ *is the "fractional part" of the real number* y.

3. Circles Ω and ω are tangent at a point P, and ω lies inside Ω. A chord AB of Ω is tangent to ω at C. The line PC meets Ω again at Q. Chords QR and QS of Ω are tangent to ω. Let I, X and Y be the incentres of triangles ABP, ABR and ABS respectively. Prove that $\angle PXI + \angle IYP = 90°$.