NST 4

Oundle, May 28 2014

1. Let Γ be the circumcircle of triangle ABC. The midpoints of the sides AB and AC are M and N respectively. The midpoint of the arc BC of Γ which does not contain A is T. The circumcircles of triangles AMT and ATN intersect the perpendicular bisectors of AC and AB at points X and Y respectively. Assume that X and Y lie inside triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

2. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right)$$

for all rational numbers x, integers a and positive integers b.

3. Let ν be an irrational positive number and m be a positive integer. An ordered pair of positive integers (a, b) is *good* if

$$a\lceil b\nu \rceil - b\lfloor a\nu \rfloor = m.$$

A good pair (a, b) is *excellent* if neither of the pairs $(a - b, b)$ nor $(a, b - a)$ is good. Prove that the number of excellent pairs is the sum of the positive divisors of m. *As usual* $\lfloor x \rfloor$ *and* $\lceil x \rceil$ *are the integers such that* $x - 1 < \lfloor x \rfloor \leq x$ *and* $x \leq \lceil x \rceil < x + 1$.