THE 3RD ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 1: FRIDAY, FEBRUARY 26, 2010, BUCHAREST

Language: English

Problem 1. For a finite non-empty set of primes P, let $m(P)$ be the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P.

(i) Show that $|P| \leq m(P)$, with equality if and only if $\min(P) > |P|$;

(ii) Show that $m(P) < (|P| + 1)(2^{|P|} - 1)$.

(The number $|P|$ is the size of the set P.)

Problem 2. For each positive integer n, find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on the closed interval $0 \leq x \leq 1$, one can find numbers $x_1, x_2, \ldots, x_n$, such that $0 \leq x_i \leq 1$, satisfying

$$\left| f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1 x_2 \cdots x_n \right| \geq C_n.$$

Problem 3. Let $A_1A_2A_3A_4$ be a convex quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ (indices are considered modulo 4, so $A_0 = A_4$, $A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Each of the three problems is worth 7 points.

Time allowed: $4\frac{1}{2}$ hours.