THE 3RD ROMANIAN MASTER OF MATHEMATICS COMPETITION
DAY 2: SATURDAY, FEBRUARY 27, 2010, BUCHAREST

Language: English

Problem 4. Determine whether there exist a polynomial \( f(x_1, x_2) \) in two variables, with integer coefficients, and two points \( A = (a_1, a_2) \) and \( B = (b_1, b_2) \) in the plane, satisfying all the following conditions:

(i) \( A \) is an integer point (i.e., \( a_1 \) and \( a_2 \) are integers);
(ii) \(|a_1 - b_1| + |a_2 - b_2| = 2010\);
(iii) \( f(n_1, n_2) > f(a_1, a_2) \), for all integer points \( (n_1, n_2) \) in the plane other than \( A \);
(iv) \( f(x_1, x_2) > f(b_1, b_2) \), for all points \( (x_1, x_2) \) in the plane other than \( B \).

Problem 5. Let \( n \) be a given positive integer. Say that a set \( K \) of points with integer coordinates in the plane is connected if for every pair of points \( R, S \in K \), there exist a positive integer \( \ell \) and a sequence \( R = T_0, T_1, \ldots, T_\ell = S \) of points in \( K \), where each \( T_i \) is distance 1 away from \( T_{i+1} \). For such a set \( K \), we define the set of vectors

\[ \Delta(K) = \{ \overrightarrow{RS} \mid R, S \in K \}. \]

What is the maximum value of \(|\Delta(K)|\) over all connected sets \( K \) of \( 2n + 1 \) points with integer coordinates in the plane?

Problem 6. Given a polynomial \( f(x) \) with rational coefficients, of degree \( d \geq 2 \), we define the sequence of sets \( f^0(Q), f^1(Q), \ldots \) by \( f^0(Q) = Q \) and \( f^{n+1}(Q) = f(f^n(Q)) \) for \( n \geq 0 \). (Given a set \( S \), we write \( f(S) \) for the set \( \{ f(x) \mid x \in S \} \).)

Let \( f^\omega(Q) = \bigcap_{n=0}^{\infty} f^n(Q) \) be the set of numbers that are in all of the sets \( f^n(Q) \). Prove that \( f^\omega(Q) \) is a finite set.

Each of the three problems is worth 7 points.
Time allowed: 4 \( \frac{1}{2} \) hours.