The 8^{th} Romanian Master of Mathematics Competition

Day 2: Saturday, February 27, 2016, Bucharest

Language: English

Problem 4. Let x and y be positive real numbers such that $x + y^{2016} \geq 1$. Prove that $x^{2016} + y > 1 - 1/100$.

Problem 5. A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω of radius R. The diagonals A_1B_2, A_2B_3, and A_3B_1 concur at X. For $i = 1, 2, 3$, let ω_i be the circle tangent to the segments XA_i and XB_i, and to the arc A_iB_i of Ω not containing other vertices of the hexagon; let r_i be the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

(b) If $R = r_1 + r_2 + r_3$, prove that the six points where the circles ω_i touch the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

Problem 6. A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-*tree* is a configuration of $n - 1$ segments, each of which has an endpoint in $\mathcal{A}$ and the other in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2 and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.