The 9th Romanian Master of Mathematics Competition

Day 2: Saturday, February 25, 2017, Bucharest

Language: English

Problem 4. In the Cartesian plane, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at distinct points A and B. The four tangents to $\mathcal{G}_1$ and $\mathcal{G}_2$ at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same axis of symmetry.

Problem 5. Fix an integer $n \geq 2$. An $n \times n$ *sieve* is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Problem 6. Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.