# The 12th Romanian Master of Mathematics Competition

Day 1: Friday, February 28, 2020, Bucharest

Language: English

**Problem 1.** Let $ABC$ be a triangle with a right angle at $C$. Let $I$ be the incentre of triangle $ABC$, and let $D$ be the foot of the altitude from $C$ to $AB$. The incircle $\omega$ of triangle $ABC$ is tangent to sides $BC$, $CA$ and $AB$ at $A_1$, $B_1$ and $C_1$, respectively. Let $E$ and $F$ be the reflections of $C$ in lines $C_1A_1$ and $C_1B_1$, respectively. Let $K$ and $L$ be the reflections of $D$ in lines $C_1A_1$ and $C_1B_1$, respectively.

Prove that the circumcircles of triangles $A_1EI$, $B_1FI$ and $C_1KL$ have a common point.

**Problem 2.** Let $N \geq 2$ be an integer, and let $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots, b_N)$ be sequences of non-negative integers. For each integer $i \notin \{1, \ldots, N\}$, let $a_i = a_k$ and $b_i = b_k$, where $k \in \{1, \ldots, N\}$ is the integer such that $i - k$ is divisible by $N$. We say $\mathbf{a}$ is $\mathbf{b}$-*harmonic* if each $a_i$ equals the following arithmetic mean:

$$a_i = \frac{1}{2b_i + 1} \sum_{s=-b_i}^{b_i} a_{i+s}.$$

Suppose that neither $\mathbf{a}$ nor $\mathbf{b}$ is a constant sequence, and that both $\mathbf{a}$ is $\mathbf{b}$-harmonic and $\mathbf{b}$ is $\mathbf{a}$-harmonic.

Prove that at least $N + 1$ of the numbers $a_1, \ldots, a_N, b_1, \ldots, b_N$ are zero.

**Problem 3.** Let $n \geq 3$ be an integer. In a country there are $n$ airports and $n$ airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and $m$ distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: $c_1$ and $c_2$; $c_2$ and $c_3$; $\ldots$; $c_{m-1}$ and $c_m$; $c_m$ and $c_1$.

Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.