The 12^{th} Romanian Master of Mathematics Competition

Day 2: Saturday, February 29, 2020, Bucharest

Language: English

Problem 4. Let $\mathbb{N}$ be the set of all positive integers. A subset A of $\mathbb{N}$ is *sum-free* if, whenever x and y are (not necessarily distinct) members of A, their sum $x + y$ does not belong to A.

Determine all surjective functions $f: \mathbb{N} \to \mathbb{N}$ such that, for each sum-free subset A of $\mathbb{N}$, the image $\{f(a): a \in A\}$ is also sum-free.

Note: a function $f: \mathbb{N} \to \mathbb{N}$ is surjective *if, for every positive integer n, there exists a positive integer m such that $f(m) = n$.*

Problem 5. A *lattice point* in the Cartesian plane is a point whose coordinates are both integers. A *lattice polygon* is a polygon all of whose vertices are lattice points.

Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Ω such that the vertices of Γ all lie on the boundary of Ω, and exactly one vertex of Ω is not a vertex of Γ.

Problem 6. For each integer $n \geq 2$, let $F(n)$ denote the greatest prime factor of n. A *strange pair* is a pair of distinct primes p and q such that there is no integer $n \geq 2$ for which $F(n)F(n+1) = pq$.

Prove that there exist infinitely many strange pairs.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.