The 13^{th} Romanian Master of Mathematics Competition

Day 1: Tuesday, October 12, 2021, Bucharest

Language: English

Problem 1. Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$.

Problem 2. Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers a_1, a_2, $\dots$, a_{20} such that, for each $k = 1, 2, \dots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k \colon k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Problem 3. A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Each of the three problems is worth 7 marks.
Time allowed $4\frac{1}{2}$ hours.