The 13^{th} Romanian Master of Mathematics Competition

Day 2: Wednesday, October 13, 2021, Bucharest

Language: English

Problem 4. Consider an integer $n \geq 2$ and write the numbers $1, 2, \ldots, n$ down on a board. A move consists in erasing any two numbers a and b, then writing down the numbers $a+b$ and $|a-b|$ on the board, and then removing repetitions (e.g., if the board contained the numbers $2, 5, 7, 8$, then one could choose the numbers $a = 5$ and $b = 7$, obtaining the board with numbers $2, 8, 12$). For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

Problem 5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and 60° rotational symmetry (that is, there is a point O such that a 60° rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2 + 3n + 1$ citizens at $3n^2 + 3n + 1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

Problem 6. Initially, a non-constant polynomial $S(x)$ with real coefficients is written down on a board. Whenever the board contains a polynomial $P(x)$, not necessarily alone, one can write down on the board any polynomial of the form $P(C+x)$ or $C+P(x)$, where C is a real constant. Moreover, if the board contains two (not necessarily distinct) polynomials $P(x)$ and $Q(x)$, one can write $P(Q(x))$ and $P(x)+Q(x)$ down on the board. No polynomial is ever erased from the board.

Given two sets of real numbers, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, a polynomial $f(x)$ with real coefficients is (A, B)-*nice* if $f(A) = B$, where $f(A) = \{f(a_i) \colon i = 1, 2, \ldots, n\}$.

Determine all polynomials $S(x)$ that can initially be written down on the board such that, for any two finite sets A and B of real numbers, with $|A| = |B|$, one can produce an (A, B)-nice polynomial in a finite number of steps.

Each of the three problems is worth 7 marks. Time allowed $4\frac{1}{2}$ hours.