The 14^{th} Romanian Master of Mathematics Competition

Day 1: Wednesday, March 1^{st}, 2023, Bucharest

Language: English

Problem 1. Determine all prime numbers p and all positive integers x and y satisfying
$$x^3 + y^3 = p(xy + p).$$

Problem 2. Fix an integer $n \geqslant 3$. Let $\mathcal{S}$ be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in $\mathcal{S}$, the triangle ABC is *nice for* AB if $\text{Area}(ABC) \leqslant \text{Area}(ABX)$ for all X in $\mathcal{S}$ different from A and B. (Note that for a segment AB there could be several nice triangles.) A triangle is *beautiful* if its vertices are all in $\mathcal{S}$ and it is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n-1)$ beautiful triangles.

Problem 3. Let $n \geqslant 2$ be an integer, and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ in the Cartesian plane, $f(x_1, y_1, \ldots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f.

(Note, for example, that the degree of the polynomial
$$g(x, y) = 4x^3y^4 + yx + x - 2$$
is 7 because $7 = 3 + 4$.)

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.