The 16^{th} Romanian Master of Mathematics Competition

Day 1: 12 February, 2025, Bucharest

Language: English

Problem 1. Let $n > 10$ be an integer, and let $A_1, A_2, \ldots, A_n$ be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j, k)$ to be the 10^{th} smallest of the distances from A_j to $A_1, A_2, \ldots, A_k$, excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j, n) \geq \frac{1}{2} f_{10}(n, n)$ for all j in the range $1 \leq j \leq n-1$.

Problem 2. Consider an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n. Is it possible for two terms of such a sequence to be equal?

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

> For any tree T with vertices $v_1, v_2, \ldots, v_n$ and any pairwise distinct complex numbers $z_1, z_2, \ldots, z_n$, there is a polynomial $P(X, Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j.

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 *because* $7 = 3 + 4$.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.