The 16^{th} Romanian Master of Mathematics Competition

Day 2: 13 February, 2025, Bucharest

Language: English

Problem 4. Let $\mathbb{Z}$ denote the set of integers, and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100}. Fix a positive integer c. Determine all functions $f\colon S \to \mathbb{Z}$ satisfying $f(xy + c) = f(x) + f(y)$ for all $x, y \in S$.

Problem 5. Let ABC be an acute triangle with $AB < AC$, and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circumcircle of triangle BOC. Circle Γ intersects line AO at points O and A', and Γ intersects the circle of radius AO with centre A at points O and F. Prove that the circle which has diameter AA', the circumcircle of triangle AFH, and Γ pass through a common point.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets $S_1, S_2, \ldots, S_k$, each of which has exactly $m + 1$ elements: one red and m blue. Let $\mathcal{F}$ be the family of all subsets T of $S_1 \cup S_2 \cup \cdots \cup S_k$ such that, for every i, the intersection $T \cap S_i$ is monochromatic. Determine the largest possible number of sets in a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that no two sets in $\mathcal{G}$ are disjoint.

A set is monochromatic if all of its elements have the same colour. In particular, the empty set is monochromatic.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.