NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Second International Selection Test

Reading, Saturday 10th May 1986

$3\frac{1}{2}$ hours

1. A, B, C, A', B', C' are six points on a circle such that the chords AA', BB', CC' meet in a point. Prove that

$$AB \,.\, B'C' \,.\, CA' = A'B' \,.\, BC \,.\, C'A. \qquad (*)$$

Is it true, conversely, that if A, B, C, A', B', C' are six points on a circle satisfying (*) then AA', BB', CC' are concurrent? Justify your answer.

2. Find, with proof, the number of different (i.e. non-congruent) triangles with sides of integer length whose perimeter is 1986 units.

3. Find, with proof, the largest real number k with the following property:

Whenever a, b, h are constants such that

$$ax^2 + 2hx + b > 0$$

for all real numbers x, then

$$a(x^2+y^2) + b(z^2+1) + h\{xz+y+k(x-yz)\} > 0$$

for all real numbers x, y, z .

4. Solve the equations

$$x^2+(y-z)^2 = a^2,$$
$$y^2+(z-x)^2 = b^2,$$
$$z^2+(x-y)^2 = c^2$$

for x, y, z, where a, b, c are given.

5. A sequence of polynomials $P_m(x,y)$, m = 0,1,2, ... in x and y is defined by $P_0(x,y) = 1$ and

$$P_m(x,y) = (x+y)(y+1)P_{m-1}(x,y+1) - y^2 P_{m-1}(x,y)$$

for m>0. Prove that each $P_m(x,y)$ is symmetric, i.e. $P_m(x,y) = P_m(y,x)$.

6. Show that the sequence of integers $[n\sqrt{2}]$, n=1,2, ... contains an infinite number of integer powers of 2. Here [x] denotes the integer part of x; for example $[3\sqrt{2}] = [4.242...]=4$.