United Kingdom Mathematics Trust

UK Mathematical Olympiad for Girls

20 September 2012

Instructions

- *Time allowed: 3 hours.*
- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top left hand corner.*
- *To accommodate candidates sitting in other timezones, please do not discuss the paper on the internet until 8am BST on Friday 21 September.*

Do not turn over until **told to do so**.

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1. The numbers a, b and c are real. Prove that at least one of the three numbers $(a+b+c)^2 - 9bc$, $(a+b+c)^2 - 9ca$ and $(a+b+c)^2 - 9ab$ is non-negative.

2. Let $S = \{a_1, a_2, \ldots, a_n\}$ where the a_i are different positive integers. The sum of the elements of each non-empty proper subset of S is not divisible by n. Show that the sum of all elements of S is divisible by n. *Note that a proper subset of S consists of some, but not all, of the elements of S.*

3. Find all positive integers m and n such that $m^2 + 8 = 3^n$.

4. Does there exist a positive integer N which is a power of 2, and a different positive integer M obtained from N by permuting its digits (in the usual base 10 representation), such that M is also a power of 2? *Note that we do not allow the base 10 representation of a positive integer to begin with* 0.

5. Consider the triangle ABC. Squares $ALKB$ and $BNMC$ are attached to two of the sides, arranged in a "folded out" configuration (so the interiors of the triangle and the two squares do not overlap one another). The squares have centres O_1 and O_2 respectively. The point D is such that $ABCD$ is a parallelogram. The point Q is the midpoint of KN, and P is the midpoint of AC.

   (a) Prove that triangles ABD and BKN are congruent.

   (b) Prove that O_1QO_2P is a square.

Time allowed: 3 hours

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

The mark allocation on Maths Olympiad papers is different from what you are used to at school. To get any marks, you need to make significant progress towards the solution. So 3 marks roughly means that you had most of the relevant ideas, but were not able to link them into a coherent proof. 8 or 9 marks means that you have solved the problem, but have made a minor calculation error or have not explained your reasoning clearly enough.

The UK MOG 2012 was marked on Sunday 30 September at St Paul's School, London, by a team of Mary Fortune, James Gazet, Jeremy King, Jonathan Lee, Gerry Leversha, Joseph Myers, Vicky Neale, Peter Neumann, Jack Shotton, Geoff Smith and Alison Zhu, who also provided the remarks and extended solutions here.

1. The numbers a, b and c are real. Prove that at least one of the three numbers $(a+b+c)^2 - 9bc$, $(a+b+c)^2 - 9ca$ and $(a+b+c)^2 - 9ab$ is non-negative.

   (Croatia 2008)

   Solution A

   It will be enough to show that the sum of the three expressions is non-negative, since the sum of three negative expressions would have to be negative. But this sum is

$$\begin{aligned}
3(a+b+c)^2 - 9(ab+bc+ca) &= 3(a^2+b^2+c^2+2ab+2bc+2ca) - 9(ab+bc+ca) \\
&= 3(a^2+b^2+c^2-ab-bc-ca) \\
&= \frac{3}{2}(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2) \\
&= \frac{3}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0
\end{aligned}$$

   since squares of real numbers are always non-negative.

Solution B

Since the expressions are symmetric in a, b and c, we may assume without loss of generality that $a \leq b \leq c$. Now $(a+b+c)^2 \geq 0$ since it is the square of a real number. In the case that $a \leq 0 \leq c$, we note that $-9ca \geq 0$ so the expression $(a+b+c)^2 - 9ca$ will be non-negative and we are finished.

In the case that $0 \leq a \leq b \leq c$, the largest of the three expressions will clearly be

$$\begin{aligned}(a+b+c)^2 - 9ab &= (a^2+b^2+c^2+2ab+2bc+2ca) - 9ab \\ &= (a^2-2ab+b^2) + (c^2-ab) + (2bc-2ab) + (2ca-2ab) \\ &= (a-b)^2 + (c^2-ab) + 2(bc-ab) + 2(ca-ab).\end{aligned}$$

But this is non-negative since the first term is the square of a real number and $c^2 \geq ab$, $bc \geq ab$ and $ca \geq ab$ in this case.

In the final case that $a \leq b \leq c \leq 0$, the largest of the three expressions will be

$$\begin{aligned}(a+b+c)^2 - 9bc &= (a^2+b^2+c^2+2ab+2bc+2ca) - 9bc \\ &= (b^2-2bc+c^2) + (a^2-bc) + (2ab-2bc) + (2ca-2bc) \\ &= (b-c)^2 + (a^2-bc) + 2(ab-bc) + 2(ca-bc).\end{aligned}$$

But this is non-negative since the first term is the square of a real number and $a^2 \geq bc$, $ab \geq bc$ and $ca \geq bc$ in this case.

Remarks

The key to proving that a quantity is non-negative is usually to show that it is the square of a real number, or the sum of more than one such square. Both solutions rely on recognising such useful identities as $a^2 - 2ab + b^2 = (a-b)^2$.

In Solution A, the idea of adding the three expressions is far from obvious, but it has the advantage of creating a single expression which is completely symmetric in a, b and c. Rewriting $a^2+b^2+c^2-ab-bc-ca$ as $\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$ is a particularly clever step if you have not seen it before.

There are some exotic methods of proving the key inequality $a^2+b^2+c^2 \geq ab+bc+ca$, such as using the AM–GM, Cauchy–Schwarz or Rearrangement Inequalities. But the details of how these are being used must be provided by the candidate, and any relevant restrictions (such as a, b and c being positive) should be observed.

The approach in Solution B was more common. Where the order of a, b and c is irrelevant, it can be helpful to choose a fixed order arbitrarily. The phrase 'without loss of generality' means that we are proving the result for all possible orders by assuming one particular order.

A clever shortcut that some candidates found was to reduce the case $a \leq b \leq c \leq 0$ to the case $0 \leq a \leq b \leq c$ by replacing a, b and c by $-c$, $-b$ and $-a$ respectively.

Trying a few values for a, b and c can be a good idea to help get a feel for the problem, but this is not sufficient for a proof. Algebra is essential here to show that the result holds for all values of a, b and c. Using algebra to prove special cases such as $a = b$ may give further insight, but this is still not sufficient for the general case.

Some candidates chose to replace b and c by new variables such as $a + x$ and $a + y$. This could be a helpful step in Solution B since x and y would be non-negative, but it destroys the symmetry in Solution A.

There are several traps when dealing with inequalities. It is vital to remember that if $X \leq Y$ then $-X \geq -Y$. If $X \leq Y$ it does not always follow that $X^2 \leq Y^2$. Also, if we establish that $X \leq Y$ and $Y \geq Z$, we can say nothing about the relationship between X and Z.

2. Let $S = \{a_1, a_2, \ldots, a_n\}$ where the a_i are different positive integers. The sum of the elements of each non-empty proper subset of S is not divisible by n. Show that the sum of all elements of S is divisible by n. *Note that a proper subset of S consists of some, but not all, of the elements of S.*

   (Slovenia 2002)

   Solution Here is a direct proof. For each i in the range 1 to n let $s_i = a_1 + a_2 + \cdots + a_i$. Now if two quantitites s_j, s_k (with $j < k$ leave the same remainder on division by n, then subtracting we see that $s_{j+1} + \cdots + s_k$ is divisible by n. Therefore we are finished unless each s_i leaves a different remainder on division by n. However, there are only n possible remainders on division by n, and so some s_i leaves remainder 0 in division by n, and therefore is divisible by n.

   Here is another way of presenting the same ideas, but this time by using a contradiction argument. Assume, for contradiction, that a counterexample S exists. Let $s_i = a_1 + a_2 + \cdots + a_i$ for $i = 1$ to n. If for different j and k with $j < k$ we have $s_k - s_j$ divisible by n, then $a_{j+1} + a_{j+2} + \cdots + a_k$ is a multiple of n which is impossible. Therefore s_1, s_2, $\ldots$, s_n leave different remainders on division by n, and so one of them must leave remainder 0, which is absurd.

   It is not important for the validity of the solution, but it is interesting to note that the configuration described can arise. For example, choose $a_i = in + 1$ for each i, so the a_i are different, and each leaves remainder 1 on division by n. Therefore the sum of any j of these numbers $(0 < j < n)$ leaves remainder $j \neq 0$ on division by n.

3. Find all positive integers m and n such that $m^2 + 8 = 3^n$.

   (South Africa, Sharp Competition in the 151st Mathematical Digest)

   Solution Suppose that m and n are positive integers satisfying $m^2 + 8 = 3^n$.

   Then $m^2 = 3^n - 8$ is odd (since 3^n is odd), so m must be odd.

   The square of any odd number leaves remainder 1 when divided by 4 (in the language of modular arithmetic, if m is odd then $m^2 \equiv 1 \pmod 4$).

   So $m^2 + 8$ also leaves remainder 1 when divided by 4.

   This means that n must be even (if n is odd, then 3^n leaves remainder 3 when divided by 4). Say that $n = 2k$, where k is also a positive integer.

Then the equation $m^2+8=3^n$ can be rewritten as $8=3^{2k}-m^2=(3^k-m)(3^k+m)$.

Since m and k are integers, so are 3^k-m and 3^k+m. Also, m and k are positive and so 3^k+m is positive, so 3^k-m must also be positive, and $3^k-m<3^k+m$. So 3^k-m and 3^k+m are positive integers whose product is 8, and we can check the possibilities.

Case 1: $3^k-m=1$, $3^k+m=8$. Then, adding these, $2\times 3^k=9$ — but this gives a value for 3^k that is not an integer, so this is not possible.

Case 2: $3^k-m=2$, $3^k+m=4$. Then, adding these and dividing by 2, we get $3^k=3$ and so $k=1$ and $m=1$. This gives $m=1$ and $n=2$, and it is clear that this really is a solution.

So the only solution is $m=1$, $n=2$.

Remarks When a question asks us to "find all solutions", this means that we should find all solutions and give a proof that we have done so. Many candidates correctly noticed that $m=1$, $n=2$ is a solution, but rather fewer had a strategy for showing that they had found them all.

The ideas of modular arithmetic are really important in mathematics, and if you aren't familiar with them then you might like to do some research. There's an NRICH article http://nrich.maths.org/4350 on the subject of modular arithmetic.

With this question, we assumed that m and n satisfied the equation $m^2+8=3^n$, and we had to find a way to draw some conclusions about m and n. Quite a lot of candidates noticed that m must be odd, and this is a useful start. Thinking about remainders on division by 4 is a way of taking this idea further, and using 4 as the modulus was a good plan both because it meant that the 8 would disappear and also because every odd square leaves remainder 1 when divided by 4.

We might well have then played around to see what happens to powers of 3 when divided by 4. This is clear in the notation of modular arithmetic, because $3^n \equiv (-1)^n \pmod 4$. (Playing around with examples can be very helpful for getting a feel for what is going on, but it then needs to be followed by a proof of any resulting assertions.) And we find that n must be even.

Once we know that n is even, we really have to write it as twice an integer, and to rewrite the resulting equation.

At this point we needed to spot the difference of two squares. Quite a few candidates asserted that the only squares that differ by 8 are 1 and 9, but not many of them backed that up with a proof. It may seem intuitively obvious, but it is important to have a rigorous justification.

Factorising a^2-b^2 as $(a-b)(a+b)$ is very often a good idea, and one should always be alert for opportunities to use this factorisation.

Candidates who thought of the factorisation were then typically able to finish off the argument pretty efficiently, as described above.

When solving a *Diophantine equation* (an equation where we seek integer solutions), one often makes use of the fact that we want *integer* solutions by thinking about factors, and this question is a good example of that.

4. Does there exist a positive integer N which is a power of 2, and a different positive integer M obtained from N by permuting its digits (in the usual base 10 representation), such that M is also a power of 2? *Note that we do not allow the base 10 representation of a positive integer to begin with* 0.

   (Iran Mathematical Olympiad, first round 1999)

   Solution We give a proof by contradiction. Suppose (for contradiction) that N is a power of 2 and that M is obtained from M by rearranging its digits, and is also a power of 2. We may assume that $M < N$ (else swap M and N). Next, since they are both powers of 2, it follows that $N = 2M$, $4M$ or $8M$. The ratio cannot be 16 or more because M and N have the same number of digits.

   Now, the remainder when a positive integer is divided by 9 is the same as when the sum of its digits is divided by 9. You can quote this as a *well-known fact*, but in case you have not seen a proof, we will supply one shortly. It follows that M and N leave the same remainder on division by 9, so their difference (M, $3M$ or $7M$) is divisible by 9. In each case, it must be that M is divisible by 9. However, M is a power of 2, so this is impossible. Our initial assumption was therefore false, so no such N exists.

   The proof is finished, but in case you have never seen a proof of the well-known fact, here it is. Suppose that the positive integer u is written $a_n a_{n-1} \dots a_1 a_0$ in base 10 notation. This is just a short way of saying that

$$u = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0.$$

   Now let $s = a_n + a_{n-1} + \cdots + a_0$ be the sum of the digits of u.

   First we show that $u - s$ is divisible by 9. Now

$$u - s = (a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_2 10 + a_1) - (a_n + a_{n-1} + \cdots + a_0)$$

$$= a_n(10^n - 1) + a_{n-1}(10^{n-1} - 1) + \cdots + a_1(10 - 1) + 0.$$

   The numbers which are of the form $10^i - 1$, for $i \geq 1$ are, in base 10 notation, 9, 99, 999, 9999, etc., and each of this is a multiple of 9 because $9 = 9 \cdot 1, 99 = 9 \cdot 11, 999 = 9 \cdot 111$ and so on. Therefore each $a_i(10^i - 1)$ is a multiple of 9, and therefore so too is their sum, so $u - s$ is divisible by 9.

   Write u and s with smallest possible remainders r_1 and r_2 on division by 9, so $u = 9k + r_1$ and $s = 9l + r_2$ where r_1, r_2 are non-negative integers which are as small as possible, so each is less than 9. Then $u - s = 9(k - l) + (r_1 - r_2)$. However, this is divisible by 9, so $r_1 - r_2$ is divisible by 9. However, $-8 \leq r_1 - r_2 \leq 8$ and so $r_1 - r_2 = 0$, and therefore $r_1 = r_2$. We have established the "fact" used in the proof above.

We could have clinched the main argument another way. We could have said "it is a well known fact that the number M and its digit sum s differ by a multiple of 9. The same is true for N, and so $N - M = (N - s) - (M - s) = N - M$ is a multiple of 9".

The special case that a number is divisible by 9 if, and only if, its digit sum is divisible by 9, seems to be better known than the version about remainders, but the proof of the more general result is almost the same as the proof of the toy version, so really the more general result is not any harder, and it is very useful. Think about the question, "what is the remainder when 2012^{2012} is divided by 9?".

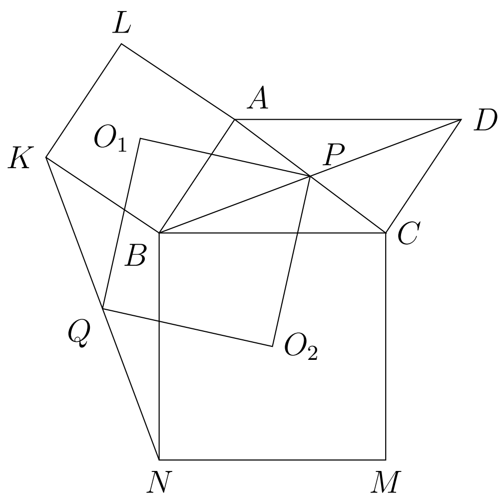
5. Consider the triangle ABC. Squares $ALKB$ and $BNMC$ are attached to two of the sides, arranged in a "folded out" configuration (so the interiors of the triangle and the two squares do not overlap one another). The squares have centres O_1 and O_2 respectively. The point D is such that $ABCD$ is a parallelogram. The point Q is the midpoint of KN, and P is the midpoint of AC.

   (a) Prove that triangles ABD and BKN are congruent.

   (b) Prove that O_1QO_2P is a square.

   (Croatia 2008)

   Solution



First we make some general remarks about how to approach geometry problems.

(a) Side angle side. We have $KB = BA$ because they are the sides of the same square. Also $BN = BC$ for the same reason, and $BC = AD$ because opposite sides of a parallelogram have the same length. Finally angle $\angle NBK$ is the supplement of angle $\angle ABC$. Now the line AB is transverse to the parallel lines AD and BC, so angles $\angle ABC$ and $\angle DAB$ are supplementary. Therefore $\angle DAB = \angle NBK$.

(b) Rotation about O_1 through a right angle effects the congruence in (a). Therefore PO_1Q is an isosceles right angled triangle. Similarly QO_2P is an isosceles right angled triangle. Putting them together we get the required square.

Remarks Note that many other solutions are possible, for example via the existence of the Varignon parallelogram with vertices at the midpoints of the sides of the quadrilateral $AKNC$. You just have to find a reason that this parallelogram is a square.

Alternatively, if you know about complex numbers, then put the diagram in the complex plane with 0 at B. Let a be be the complex number corresponding to A and c be the complex number corresponding to C. Now the complex number corresponding to P is $\frac{1}{2}(a+c)$. The the complex number corresponding to Q is $\frac{1}{2}i(a-c)$. The complex number corresponding to O_1 is $\frac{1}{2}a(1+i)$. The complex number corresponding to O_2 is $\frac{1}{2}c(1-i)$. The fact that we have a square on our hands is now a routine calculation, for example by showing that the complex numbers corresponding to its four sides (taken in anticlockwise order, and oriented anticlockwise) can be generated from one of them by repeated multiplication by i. An argument by tiling should also be possible (details only wafted in the reader's general direction): take infinitely many (or at least a few) of copies of the two squares and triangle ABC, and fit them together to tile the plane (or a portion of it) in the natural way. Then argue about rotational symmetries of the pattern.

Here is some extra advice from another marker who is also a very experienced schoolteacher:

Perhaps it is worth beginning by emphasising generality. Don't begin with an equilateral or a right-angled isosceles triangle. Such attempts rarely got any credit. Make sure that the letters follow each other in order around the figure and make sure that the diagram is correct.

A proof is not a random enumeration of facts. A candidates who simply lists many 'facts' (even if they are all true) and then expects the marker to choose the relevant two or three to fit into a proof should receive no credit. Draw an accurate diagram please, and make sure that you label different points with different letters, and that your proof relates to this diagram.

For part (a), give the correct congruence condition (SAS) and remember that the angle must be between the sides. Once you decide that is where you are going, the argument quickly reduces to showing that two angles are equal. Do not evaluate every angle in sight; take a look at what you want to prove, and then focus on important angles. It is quite acceptable to use transformational methods, but if you do so you must state the transformation precisely (rotation, centre, angle, sense) and then be sure that you have all the necessary evidence that it is relevant.

For part (b), the midpoint theorem is helpful. This says that in a triangle ABC, with B_1 and C_1 the midpoints of CA and AB respectively, the line segment C_1B_1 is parallel to BC and half its length. This leads on the the theory of the Varignon parallelogram (look it up). That very soon establishes that O_1QO_2P is a rhombus.

Hence all you have to do is to show it has one right-angle, for that will make it into a square. A rotational argument very useful here. Alternatively, two isosceles right-angled triangles on the same base form a square, since the base angles of each triangle ar 45° so we have a rectangle. The common diagonal is the length of any side multiplied by $\sqrt{2}$, so the rectangle is a square.

There were few solutions using complex numbers or coordinate methods. There were some attempts at vectors, some of which worked (although interestingly one successful solution abandoned vectors at the crucial last step and went back to rotations). So it is perhaps not worth saying much about these; those who managed to use them properly knew what they were doing. Perhaps one ought only to say that if you are going to embark on coordinates you ought to choose your origin, axes and scale carefully; similarly with a vector approach, and it is worth knowing that the column vectors $\binom{a}{b}$ and $\binom{-b}{a}$ are perpendicular. If you are going to use complex numbers, you will use the fact that multiplication by i effects an anticlockwise rotation in the usual Argand diagram, so if you want a clockwise rotation you will need to use $-i$. Spiral similarities are in the FP3 section of A-level, of course.

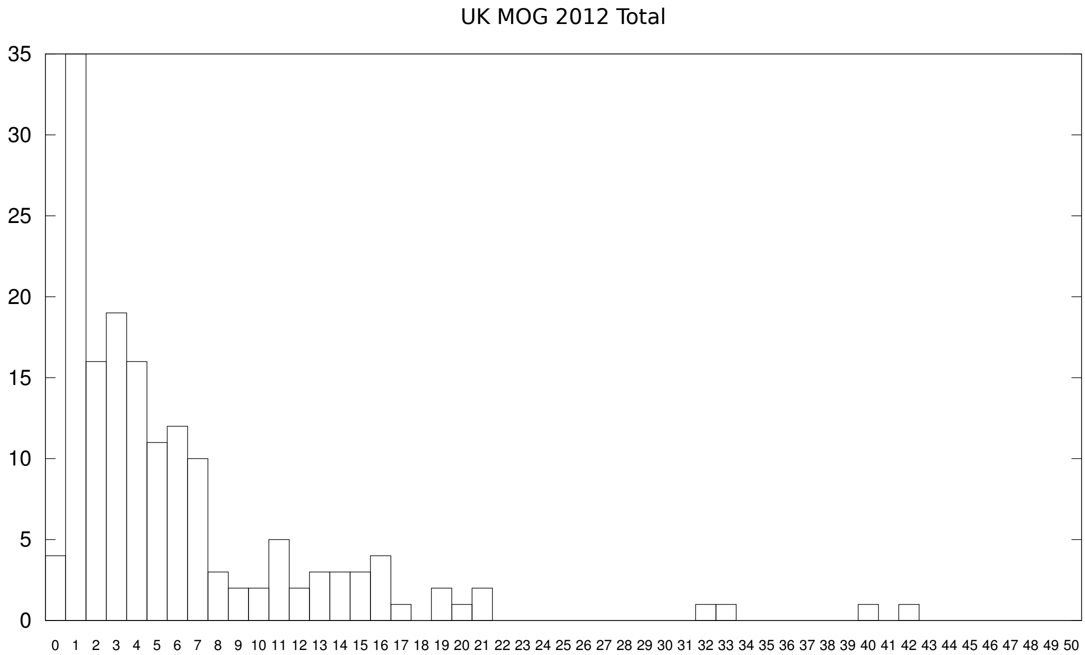
Both texts "Crossing the Bridge" and "Plane Euclidean Geometry" are available from UKMT, and will help you solve geometry problems if you use the books properly (do all the exercises).

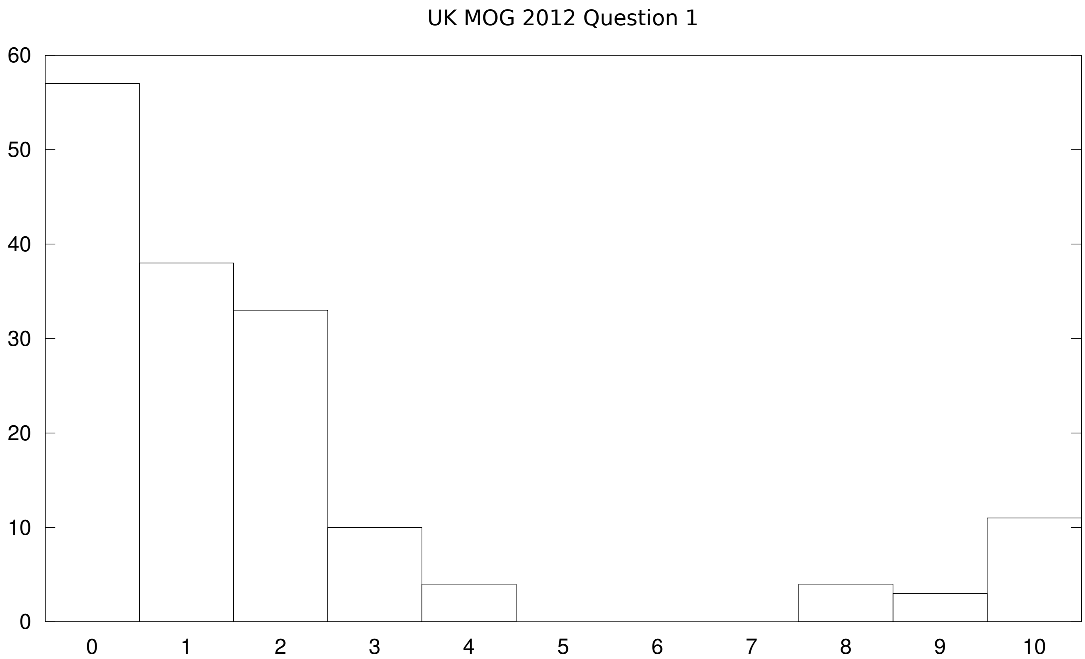
UK Mathematical Olympiad for Girls

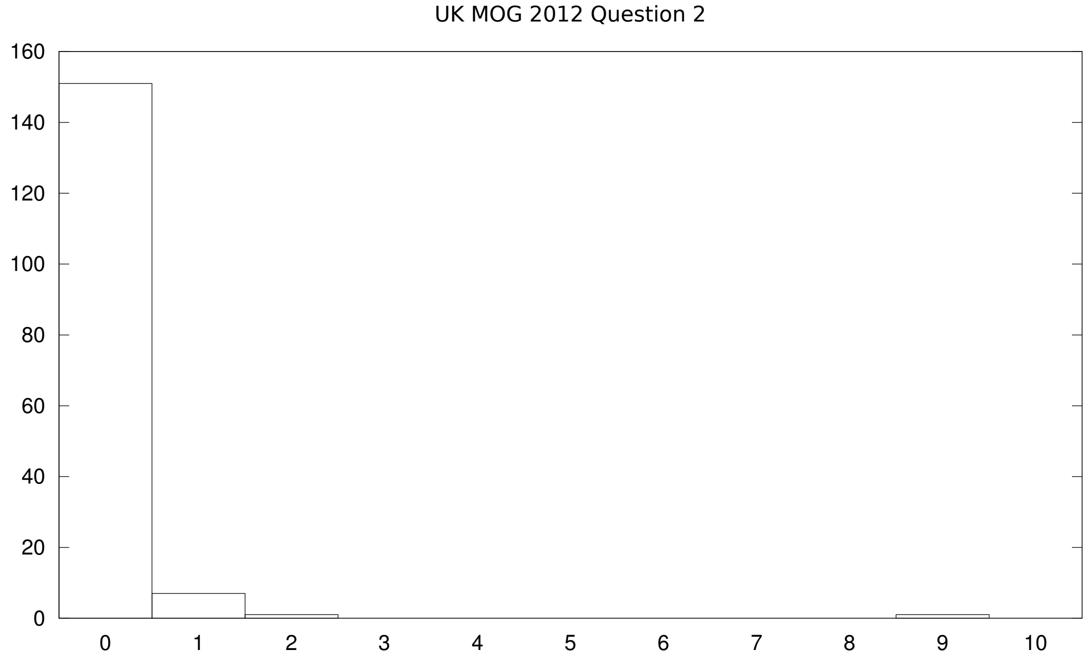
20 September 2012

Statistics of results

247 candidates entered UK MOG 2012, of whom 160 returned scripts. Statistics of the results of those 160 are shown below.






UK MOG 2012 Question 2
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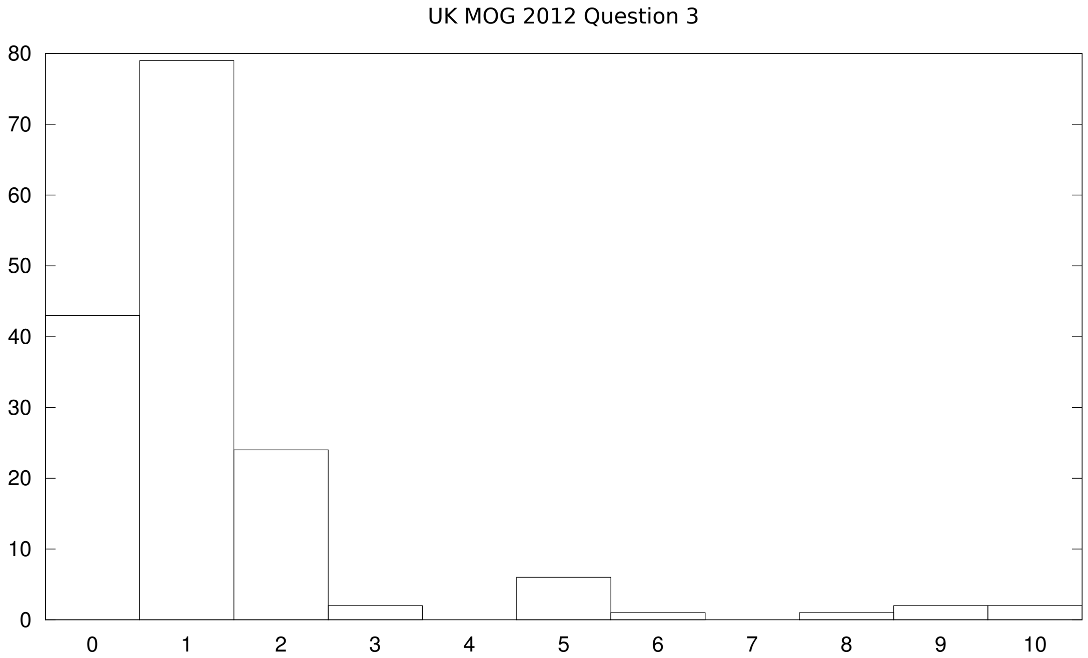
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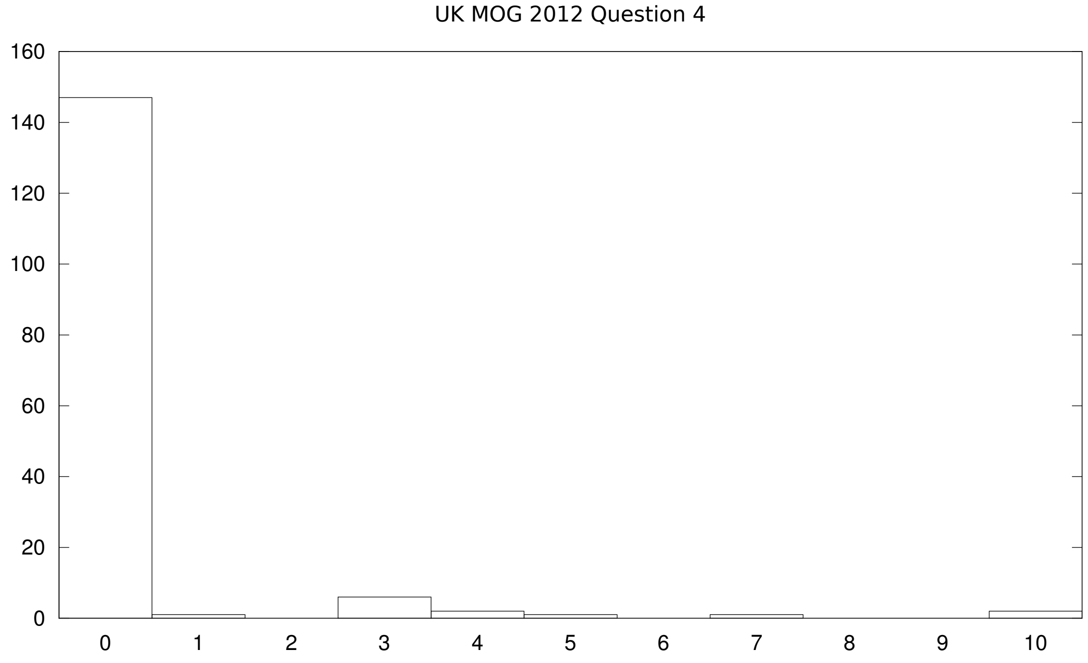
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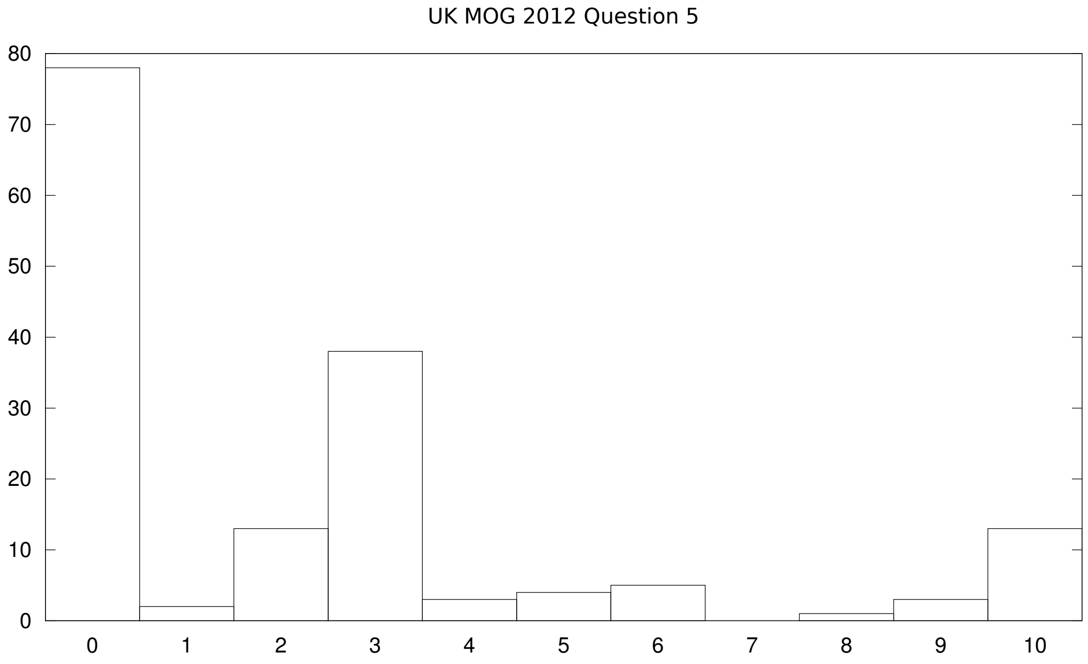
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