United Kingdom Mathematics Trust

UK Mathematical Olympiad for Girls

20 September 2012

Instructions

- *Time allowed: 3 hours.*
- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top left hand corner.*
- *To accommodate candidates sitting in other timezones, please do not discuss the paper on the internet until 8am BST on Friday 21 September.*

Do not turn over until **told to do so**.

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1. The numbers a, b and c are real. Prove that at least one of the three numbers $(a+b+c)^2 - 9bc$, $(a+b+c)^2 - 9ca$ and $(a+b+c)^2 - 9ab$ is non-negative.

2. Let $S = \{a_1, a_2, \ldots, a_n\}$ where the a_i are different positive integers. The sum of the elements of each non-empty proper subset of S is not divisible by n. Show that the sum of all elements of S is divisible by n. *Note that a proper subset of S consists of some, but not all, of the elements of S.*

3. Find all positive integers m and n such that $m^2 + 8 = 3^n$.

4. Does there exist a positive integer N which is a power of 2, and a different positive integer M obtained from N by permuting its digits (in the usual base 10 representation), such that M is also a power of 2? *Note that we do not allow the base 10 representation of a positive integer to begin with* 0.

5. Consider the triangle ABC. Squares $ALKB$ and $BNMC$ are attached to two of the sides, arranged in a "folded out" configuration (so the interiors of the triangle and the two squares do not overlap one another). The squares have centres O_1 and O_2 respectively. The point D is such that $ABCD$ is a parallelogram. The point Q is the midpoint of KN, and P is the midpoint of AC.

   (a) Prove that triangles ABD and BKN are congruent.

   (b) Prove that O_1QO_2P is a square.

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