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Mathematical Olympiad for Girls 2013

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has some sort of overall strategy or not. An answer which is essentially a solution, but might contain either errors of calculation, flaws in logic, omission of cases or technical faults, will be marked on a '10 minus' basis. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which shows no sign of being a genuine solution is marked on a '0 plus' basis; up to 3 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

This year a new format was adopted for the Mathematical Olympiad for Girls, with some questions split into two parts. The purpose of the first part is to introduce results or ideas needed to answer the second part.

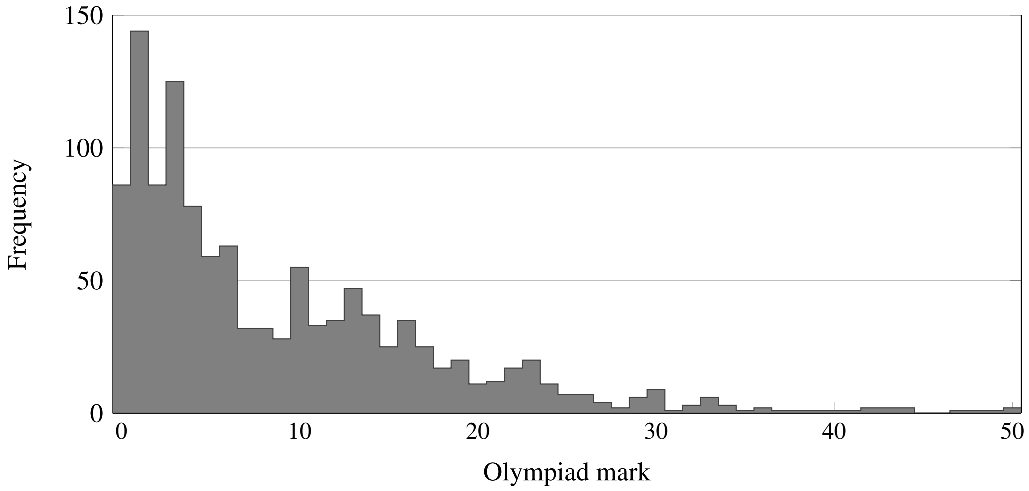
We were pleased that most candidates had a reasonable attempt at all the questions. Many completed part (a) of more than one question. It was also pleasing to see a lot of candidates making a decent attempt to explain and justify their solutions: they grasped that a single numerical answer would not suffice.

Some candidates got involved in long calculations in Questions 1 and 3, which would have been time-consuming. Most Olympiad questions are designed to be done without excessive calculation, so candidates should be encouraged to look for more elegant approaches, usually using algebra. When a long calculation seems like the only possible option, all details must be shown. Too often candidates claim that they have checked all the possibilities, but we need to see some evidence of this!

One of the most common mistakes that candidates made was trying to argue from special cases, rather than realising that some generality was needed. This was most apparent in Question 2, where many candidates considered a rectangle rather than a general quadrilateral, and in Question 4, where many only explained why certain special paths were not possible.

The 2013 Mathematical Olympiad for Girls attracted 1201 entries. The scripts were marked on Sunday 6 October at Murray Edwards College, Cambridge by a team of James Aaronson, Ross Atkins, Benjamin Barrett, Natalie Behague, Andrew Carlotti, Andrea Chlebikova, Philip Coggins, James Cranch, Rosie Cretney, Susan Cubbon, Matthew Dawes, Elena Dulskyte, Paul Fannon, Richard Freeland, Adam P. Goucher, Jo Harbour, Maria Holdcroft, Andrew Jobbings, Vesna Kadelburg, Josh Lam, David Mestel, Joseph Myers, Vicky Neale, Peter Neumann, Sylvia Neumann, Martin Orr, Preeyan Parmar, David Phillips, Aled Walker and Alison Zhu.

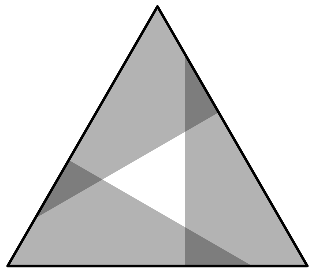
Mark distribution



Question 1

The diagram shows three identical overlapping *right-angled* triangles, made of coloured glass, placed inside an equilateral triangle, one in each corner. The total area covered twice (dark grey) is equal to the area left uncovered (white).

What fraction of the area of the equilateral triangle does one glass triangle cover?



It is possible to solve this problem by calculating the sides of the dark grey area and the glass triangles in terms of the side of the equilateral triangle. To do this, we need to write several equations relating the lengths in all the different triangles (large, glass, dark and white). One way to simplify the calculations slightly is to set the side of the equilateral triangle equal to 1 unit, as we are not interested in the actual areas, only in their ratios. Several candidates did this successfully, but most of those who tried either couldn't produce enough equations, or were not able to deal with expressions involving surds. Many candidates derived the formula for the area of the equilateral triangle; this is certainly needed, but did not score any marks because it is a standard application of trigonometry.

Luckily, it turns out that we can find the ratio of areas without calculating the sides first. In fact, it is not important that the shapes are triangles; the answer would be the same if we used any three equal shapes that overlap in pairs. A majority of the candidates adopted this approach and there were many successful solutions. Overall, around one-third of the candidates who attempted this question scored full or nearly full marks.

Here is one possible argument: As the dark grey area is covered twice, if we "take off" one layer and use it to cover the white area, then the whole equilateral triangle will be covered once. This means that the glass from all three glass triangles exactly covers the equilateral triangle, so one glass triangle covers one-third of the equilateral triangle. But is this explanation mathematically rigorous?

The above argument could in fact score full marks, but it had to be clearly written. In particular, it needs to be explicitly stated that the glass can be rearranged so that the whole triangle is covered once. A much safer way to make the argument rigorous is to give labels to important quantities and write some equations. Candidates who did this generally scored 9 or 10 marks.

Solution 10 marks

METHOD 1

Denote the whole area by E, the area of one glass triangle by T, the white area by W, one light grey part by A and one dark grey part by B.

From the given information, we have $W = 3B$.

From the diagram, we have $E = W + 3A + 3B$ and $T = A + 2B$. Using the given information, we get $E = 3A + 6B$, and now we see that $T = \frac{1}{3}E$.

So one glass triangle covers one third of the area of the equilateral triangle.

METHOD 2

Let E be the whole area, T the area of one glass triangle, D the dark grey area, and W the white area.

Since each dark grey area is covered twice, $E = 3T - D + W$.

But from the information given $D = W$, so $T = \frac{1}{3}E$.

Question 2

In triangle ABC, the median from A is the line AM, where M is the midpoint of the side BC. In any triangle, the three medians intersect at the point called the centroid, which divides each median in the ratio $2:1$.

In the convex quadrilateral $ABCD$, the points A', B', C' and D' are the centroids of the triangles BCD, CDA, DAB and ABC, respectively.

(a) By considering the triangle MCD, where M is the midpoint of AB, prove that $C'D'$ is parallel to DC and that $C'D' = \frac{1}{3}DC$.

(b) Prove that the quadrilaterals $ABCD$ and $A'B'C'D'$ are similar.

When a diagram is not given in the question it is always worth drawing one, if only to have something you can refer to in your explanation. The purpose of a diagram is to convey information, so be sure to draw diagrams clearly, don't make them too small, and label them carefully. Here you need to be aware of a labelling convention: referring to a quadrilateral as $ABCD$ means that the vertices A, B, C and D appear *in that order* as you progress around the quadrilateral. It may also be useful to draw a new diagram once you have decided which points you need to consider. For example, to answer part (a) you only need M, C' and D', not all the medians and centroids.

The term *convex* in the statement of the question seems to have been a source of some confusion, with some candidates placing vertex D inside the triangle ABC. The result in question in fact holds even when the quadrilateral is not convex. The restriction to convex quadrilaterals was meant to avoid the need for several different diagrams.

This was by far the least popular question, and a majority of those who attempted it made little or no progress. Rather too many candidates fell down because they started with a poor diagram. In particular, it is a bad idea to draw $ABCD$ as a rectangle, which is too special to represent a *general* quadrilateral: the danger is that you are tempted to draw conclusions that are not true for a more general figure.

Other candidates lost marks because they were unable to write down a careful proof that the two triangles MCD and $MC'D'$ are similar and then draw appropriate conclusions. What is required to prove two triangles similar, and what conclusions follow? There are various "tests" for two triangles to be similar; the test you need here is for two pairs of sides to be in the same ratio and the angles between them (the *included* angles) to be equal. Once you have proved two triangles are similar, you can deduce that *all* pairs of sides are in the same ratio, and *every* angle of one triangle is equal to the corresponding angle of the other triangle.

A few candidates proved part (a) using vectors. The proof given in Method 2 below uses the result for the position vector of the centroid of a triangle—it is the 'mean' of the position vectors of the vertices. You may derive this result from the information about the median and the centroid given in the question. Note that quoting the result means that this method does not explicitly use triangle MCD, as referred to in the question; this was not penalised.

To answer part (b) you need to know what it means for two quadrilaterals to be similar. For two triangles to be similar, it is enough that all corresponding angles are equal. This is not a sufficient condition for quadrilaterals; for example, two rectangles have equal angles, but they are not necessarily similar. Having all the sides in the same ratio is also not enough - think of a rhombus and a square, for example.

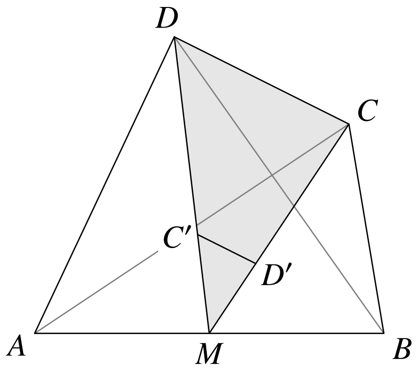
It is helpful to recall the definition of similar figures. Loosely speaking, two figures are similar if they are of the same shape, but of different sizes. More precisely, two figures are similar if all pairs of corresponding sides are in the same ratio, *and* all pairs of corresponding angles are equal. Thus in order to prove that two quadrilaterals are similar it is necessary to show that:

(i) pairs of corresponding sides are in the same ratio;
(ii) pairs of corresponding angles are equal.

Several candidates claimed that, since the sides of the two quadrilaterals are in the ratio 3 : 1, one must be an enlargement of the other. You should remember than enlargement is a geometrical transformation which is defined by a scale factor and a centre of enlargement. Two figures can be similar without being related by an enlargement. In this example, a centre of enlargement does in fact exist, but it is not any of the points mentioned in the question.

The markers felt that part (a) contained the bulk of the mathematical work needed to answer the question, so made 7 marks available for that part.

Solution to part (a) 7 marks



METHOD 1

We are given that M is the midpoint of AB, so that CM is a median of triangle ABC. Since D' is the centroid of triangle ABC, it lies on CM, and $CD' : D'M = 2 : 1$. Similarly, $DC' : C'M = 2 : 1$. This means that $MD' = \frac{1}{3}MC$ and $MC' = \frac{1}{3}MD$, so the triangles MCD and $MD'C'$ have two pairs of sides in the same ratio, namely $3 : 1$. The angle CMD is also common to both triangles. Hence the triangles are similar (they have two pairs of sides in the same ratio and the angles between those sides are equal).

It follows that the third sides are also in the ratio $3 : 1$, so that $C'D' = \frac{1}{3}DC$, as required.

Furthermore, corresponding angles in the two triangles are equal, so that $\angle MD'C' = \angle MCD$. We conclude that $C'D'$ and DC are parallel.

METHOD 2

Let the position vectors of the vertices be $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$, and let the position vectors of C' and D' be $\boldsymbol{c'}$ and $\boldsymbol{d'}$.

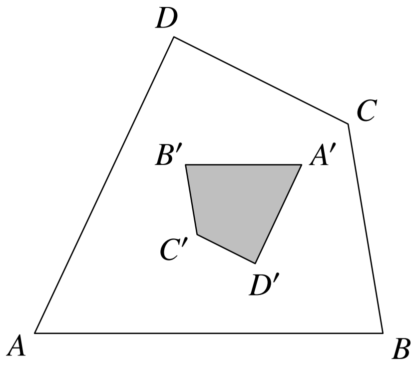
Then, since C' is the centroid of triangle ABD, $\boldsymbol{c'} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d})$. Similarly, $\boldsymbol{d'} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$.

Hence we have

$$\begin{aligned}\boldsymbol{C'D'} &= \boldsymbol{d'} - \boldsymbol{c'} \\ &= \tfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) - \tfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d}) \\ &= \tfrac{1}{3}(\boldsymbol{c} - \boldsymbol{d}) \\ &= \tfrac{1}{3}\boldsymbol{DC}.\end{aligned}$$

Therefore $C'D'$ is parallel to DC and $C'D' = \frac{1}{3}DC$.

Solution to part (b) 3 marks



We proved in part (a) that $D'C'$ is parallel to CD. Similarly, $D'A'$ is parallel to AD. Hence $\angle A'D'C' = \angle ADC$. It can be shown analogously that the other corresponding angles in the two quadrilaterals are equal.

Also from part (a), the corresponding sides of the two quadrilaterals are all in the ratio 3 : 1.

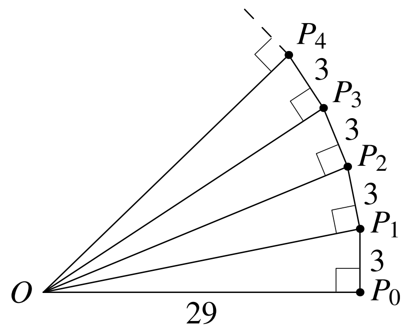
Hence the two quadrilaterals are similar.

Question 3

(a) Find all positive integers a and b for which $a^2 - b^2 = 18$.

(b) The diagram shows a sequence of points P_0, P_1, P_2, P_3, P_4, ..., which spirals out around the point O. For any point P in the sequence, the line segment joining P to the next point is perpendicular to OP and has length 3. The distance from P_0 to O is 29.

What is the next value of n for which the distance from P_n to O is an integer?



Many candidates did lengthy calculations in both parts of this question. This is an acceptable method, provided your calculations are without error and you can prove that you have systematically checked all relevant cases. For example, those who simply checked a few pairs of squares in part (a) did not score any marks. However, it is possible to argue that $a \geq 5$ because $a^2 > 18$, and also that $a \leq 9$ because differences between consecutive squares increase and $10^2 - 9^2 > 18$. Hence there are only five possible values of a to check.

If you are trying to check all possible cases, you have to show all your calculations. Markers can only give marks for what you have written down. In part (b) in particular, several candidates claimed that they calculated OP_n until they found that OP_{35} was an integer. In such a situation, markers cannot be sure that all the calculations were correct, unless they are shown explicitly.

Long calculations can be avoided if you use the difference of two squares. Moreover, the method can be used with much larger numbers. For example, Method 2 can be readily adapted to prove that there are no integer solutions of the equation $a^2 - b^2 = 174$.

In part (b), many candidates found the correct expression for OP_n but then didn't have a good plan for what to do with it. The difference of two squares from part (a) should have been a helpful hint. The situation is slightly more complicated, because after factorising we get $(OP_n - 29)(OP_n + 29) = 9n$, and since we don't know the value of n we cannot write out all factors of $9n$. However, we can think about factors of 9. One possibility is that both the factors $(K - 29)$ and $(K + 29)$ are multiples of 3. But they differ by 58, which is not a multiple of 3, so this situation is impossible. It follows that one of these factors is a multiple of 9. (Notice that they cannot both be multiples of 9.)

The expression for OP_n can be found by considering the first few points:

$$\begin{aligned} OP_1^2 &= 29^2 + 3^2 \\ OP_2^2 &= OP_1^2 + 3^2 = 29^2 + 2 \times 3^2 \\ OP_3^2 &= OP_2^2 + 3^2 = 29^2 + 3 \times 3^2 \end{aligned}$$

Each length is obtained from the previous one by using Pythagoras, so $OP_n^2 = OP_{n-1}^2 + 9$. Therefore $OP_n^2 = 29^2 + n \times 3^2$. If we were being really careful, then we would use mathematical induction to prove this assertion. However, this was not required to score full marks.

Solution to part (a) 3 marks

METHOD 1

Notice that the left-hand side is a difference of two squares and hence can be factorised as $(a-b)(a+b)$. We are looking for integer solutions, so that $a-b$ and $a+b$ have to be factors of 18. Since a and b are positive, and $a^2 > b^2$, it follows that $a-b$ and $a+b$ are both positive, and $a+b > a-b$. Hence we need to consider only the following three possibilities:

$$\begin{aligned} & a-b=1 \\ \text{and} \quad & a+b=18; \end{aligned} \quad \text{or} \quad \begin{aligned} & a-b=2 \\ \text{and} \quad & a+b=9; \end{aligned} \quad \text{or} \quad \begin{aligned} & a-b=3 \\ \text{and} \quad & a+b=6. \end{aligned}$$

Solving each pair of simultaneous equations, we get

$$a=\tfrac{19}{2},\ b=\tfrac{17}{2} \quad \text{or} \quad a=\tfrac{11}{2},\ b=\tfrac{7}{2} \quad \text{or} \quad a=\tfrac{9}{2},\ b=\tfrac{3}{2}.$$

Since none of the solutions are integers, the original equation has no solutions when a and b are positive integers.

METHOD 2

Notice that the left-hand side is a difference of two squares and hence can be factorised as $(a-b)(a+b)$. Now $a-b$ and $a+b$ differ by an even integer, namely $2b$, so they are either both odd or both even. Since their product is 18 they are both even. But then their product is a multiple of 4, which 18 is not. This is a contradiction, so we deduce that the given equation has no solutions when a and b are positive integers.

Solution to part (b) 7 marks

We have $OP_1^2 = 29^2 + 3^2$ and $OP_n^2 = OP_{n-1}^2 + 3^2$, so $OP_n^2 = 29^2 + n \times 3^2$.

Therefore

$$OP_n^2 - 29^2 = 9n.$$

Write K for OP_n. Then K is a positive integer and, after we factorise the left-hand side, our equation becomes

$$(K-29)(K+29) = 9n.$$

Now 58 is not a multiple of 3, so $K-29$ and $K+29$ cannot both be multiples of 3. Therefore either 9 divides $K-29$ or 9 divides $K+29$. We also know that $K > 29$, because $OP_n > OP_0 = 29$.

Suppose that 9 divides $K-29$. In this case the least possible value of K is 38, leading to $n = 67$.

Suppose instead that 9 divides $K+29$. In that case the least possible value of K is 34, leading to $n = 35$.

So the next value of n for which OP_n is an integer is 35.

Question 4

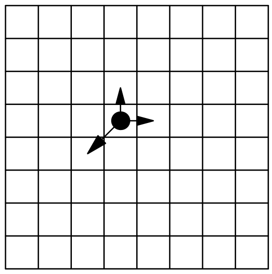
(a) An ant can move from any square on an 8×8 chessboard to an adjacent square. (Two squares are adjacent if they share a side).

The ant starts in the top left corner and visits each square exactly once. Prove that it is impossible for the ant to finish in the bottom right corner.

[*You may find it helpful to consider the chessboard colouring.*]

(b) A ladybird can move one square up, one square to the right, or one square diagonally down and left, as shown in the diagram, and cannot leave the board.

Is it possible for the ladybird to start in the bottom left corner of an 8×8 board, visit every square exactly once, and return to the bottom left corner?



This, along with Question 1, was the most popular question. It was pleasing to see that around half of the candidates who attempted it produced a successful solution to part (a). You should remember that every step of your argument needs to be clearly explained. For example, several candidates lost marks because they didn't explicitly state that the colours of adjacent squares on the ant's path alternate.

Almost all unsuccessful solutions tried to show that the required route is impossible by considering some specific examples. Although examples are a good way to start, they cannot be used to prove the general result unless you can ensure that you have tried *all* possible examples - not really feasible in this problem!

If you found that looking at the colour of the squares was useful in part (a), you may have tried a similar strategy in part (b). However, if we just colour the squares black and white, then the up and right moves are onto a different colour, but the diagonal move is onto the same colour. Since there are three types of move, it seems sensible to use three colours, and ensure that every move changes the colour of the square. This can be achieve by colouring the board in diagonal stripes, as shown in Method 1 below.

Alternatively, if you thought about even and odd squares in part (a), you may have guessed that the key to part (b) is thinking about multiples of 3. This leads to Method 2, which was the approach taken by a large majority of successful candidates.

Those who only considered certain special paths in part (a) were equally unsuccessful in part (b). However, somewhat surprisingly, there is an argument which starts by considering what must happen in corners, presented in Method 3. The corners cannot be connected without the path crossing over itself. The crucial insight is that any path that does cross over itself must visit some square more than once. Note that this would not be the case for a figure that can move along both diagonals (such as, for example, the chess king), because the crossover could happen on two diagonal moves. But the ladybird cannot do this, as it only moves along one diagonal.

Solution to part (a) 3 marks

Consider the colours of the squares on the board.

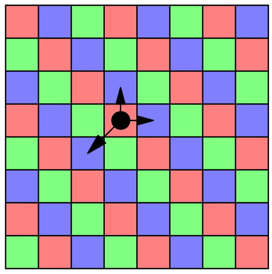
Each move changes the colour and there are 63 moves, so the start and end squares have different colours.

But the top left corner and the bottom right corner have the same colour. Hence the bottom right corner cannot be the final square.

Solution to part (b) 7 marks

Method 1

Consider a colouring with three colours (red, blue, green), in which all the squares in a diagonal stripe (running from top left to bottom right) have the same colour, but squares in adjacent stripes have different colours (see diagram).



Then the ladybird always moves from green to red, from red to blue and from blue to green. To return to the starting square (which is green) she needs to make 64 moves. But the moves which end on a green square are the moves numbered 3, 6, 9, and so on. Since 64 is not a multiple of 3, the final move cannot be onto a green square.

Method 2

Suppose that the ladybird makes a tour starting and ending in the bottom left corner.

Let u, r and d be the numbers of each type of move (up, right and diagonal, respectively). Because she returns to the starting point the number of moves right must equal the number of moves left. But the only moves left are the diagonal ones, so $r = d$. Similarly, $u = d$.

Hence the total number of moves is $u + r + d = 3d$ which cannot equal 64, and thus the required tour is impossible.

Method 3

The sequence of moves is equivalent to a path connecting the centres of squares. Such a path cannot cross over itself without passing through a square more than once.

Consider the square with centre L in the top left corner. The only possible moves are $K \to L \to M$, as shown in Figure 1.

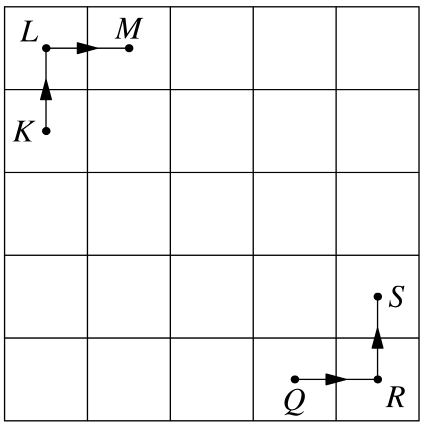


Figure 1

Now consider the square with centre R in the bottom right corner. The only possible moves are $Q \to R \to S$, as shown.

However, once we've connected S to K, we will find that Q and M are on opposite sides of the path between them. Since it can't go around the ends, this means that the path from Q to M will necessarily cross over the path from S to K, which is not possible.

Question 5

(a) Find an integer solution of the equation $x^3 + 6x - 20 = 0$ and prove that the equation has no other real solutions.

(b) Let x be $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.
Prove that x is equal to 2.

Although many candidates correctly found an integer solution to the equation in part (a) by simply trying some small values of x, a considerable number tried to apply the *quadratic* formula to this *cubic* equation. It may be an interesting exercise to investigate for which cubics of the form $x^3 + bx + c = 0$ the quadratic formula does in fact give the correct solutions.

A useful observation is that any integer solution of this equation has to be a factor of 20. This is because, if we write the equation in factorised form, the product of the constant terms will be -20. This restricts the number of integers we need to try.

It is also important to understand the difference between the instructions to “find an integer solution” and to “solve the equation”. The former simply involves finding a number that works, so guessing is allowed (as long as you check that the number does work). The latter requires a procedure that finds all the solutions, and shows that there are no others. Note that in part (a) we actually end up solving the equation, as we have found the only real solution.

In arguing that there are no solutions other than $x = 2$, many candidates made a mistake of considering only integers. Furthermore, many argued that “since $x = 1$ is too small and $x = 3$ is too big, there are no other solutions”. This argument is only valid if the expression $x^3 + 6x - 20$ always increases with x. This is in fact true, and it can be proved either by considering its derivative, or by noticing that, since both x^3 and $6x$ are increasing functions, then so is their sum. However, we needed to see the proof to award full marks.

For part (b) you needed to show that x satisfies the equation from part (a). Many candidates seemed to be intimidated by the complicated expression with cube roots, but most of those who attempted this part tried to find an expression for x^3. Unfortunately, many hastily did this by cubing both terms separately, essentially trying to use $(a - b)^3 = a^3 - b^3$, which is of course not true (except for some special values of a and b — can you find all of them?). The correct expansion is $(a - b)^3 = a^3 - 3a^2b + 3ab^3 - b^3$; you may know that this is called the binomial expansion.

The terms a^2b and ab^2 require some thought. For example, using the ‘difference of two squares’ factorisation, you can write

$$\begin{aligned}\left(\sqrt{108} + 10\right)^2\left(\sqrt{108} - 10\right) &= \left(\sqrt{108}^2 - 10^2\right)\left(\sqrt{108} + 10\right)\\ &= 8\left(\sqrt{108} + 10\right).\end{aligned}$$

However, we can make the working neater by using the fact that $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

Once you have shown that x satisfies the equation from part (a), you can conclude that $x = 2$, but only because you have shown that the equation has only one real solution. You needed to mention this explicitly to get full marks.

Solution to part (a) 3 marks

A direct check shows that $x = 2$ satisfies the equation. Therefore the cubic expression can be factorised as $(x-2)(x^2+bx+c)$. Comparing coefficients gives $x^3+6x-20 = (x-2)(x^2+2x+10)$.

Hence any other solution of the original equation satisfies $x^2 + 2x + 10 = 0$. But the discriminant of this quadratic is $2^2 - 4 \times 1 \times 10$, which is negative, so there are no other real solutions.

Solution to part (b) 7 marks

Let $a = \sqrt[3]{\sqrt{108} + 10}$ and $b = \sqrt[3]{\sqrt{108} - 10}$. Then

$$\begin{aligned} a - b &= x, \\ a^3 - b^3 &= \left(\sqrt{108} + 10\right) - \left(\sqrt{108} - 10\right) = 20 \\ \text{and} \quad ab &= \sqrt[3]{\left(\sqrt{108} + 10\right)\left(\sqrt{108} - 10\right)} = \sqrt[3]{108 - 10^2} = 2. \end{aligned}$$

Hence $x^3 = (a - b)^3 = a^3 - b^3 - 3ab(a - b) = 20 - 6x$, which is equivalent to $x^3 + 6x - 20 = 0$.

Since we have proved in part (a) that this equation has only one real solution, it follows that $x = 2$.