Mathematical Olympiad for Girls

Tuesday 29th September 2015

Organised by the United Kingdom Mathematics Trust

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. Some of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

The mark allocation on Mathematics Olympiad papers is different from what you are used to at school. To get any marks, you need to make significant progress towards the solution. This is why the rubric encourages candidates to try to finish whole questions rather than attempting lots of disconnected parts.

Each question is marked out of 10.

3 or 4 marks roughly means that you had most of the relevant ideas, but were not able to link them into a coherent proof.

8 or 9 marks means that you have solved the problem, but have made a minor calculation error or have not explained your reasoning clearly enough. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap?

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1. (a) Expand and simplify $(a - b)(a^2 + ab + b^2)$.

(b) Find the value of

$$\frac{2016^3 + 2015^3}{2016^2 - 2015^2}.$$

Solution

Commentary

The result in part (a) is called the difference of two cubes. Notice that part (b) involves a sum of two cubes in the numerator of the fraction. You will probably need to adapt the result of part (a) in order to use it in part (b).

One possibility in part (b) would be to do some calculations. However, this will be very time-consuming, so let's try to find a quicker approach. We might try to simplify the fraction by factorising both the numerator and the denominator and looking for common factors.

To avoid writing out long numbers (and the risk of making numerical errors) it is a good idea to rewrite the expression using algebraic symbols. We will substitute the numbers back in at the end. This means that we are trying to simplify the fraction

$$\frac{m^3 + n^3}{m^2 - n^2},$$

where $m = 2016$ and $n = 2015$.

The denominator factorises as $(m-n)(m+n)$. (You might know this as the "difference of two squares".) In our case, we know that $m - n = 1$, so in fact the denominator is just $m + n$.

This is where part (a) can give us a useful suggestion. We know from (a) how to factorise $a^3 - b^3$. Can we do something similar to factorise $a^3 + b^3$?

There are various ways to see how to adapt part (a), including experimentation with changing signs. We find that

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

This will be really helpful in simplifying the fraction, and then we can substitute the numbers back in.

The final expression still looks a little long to evaluate. Some clever use of factorising can simplify the calculation a little.

(a) Expanding out the brackets and simplifying, we find that

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3.\end{aligned}$$

(b) Let $m = 2016$ and $n = 2015$.

Using the factorisation

$$m^3 + n^3 = (m + n)(m^2 - mn + n^2),$$

and the difference of two squares $m^2 - n^2 = (m - n)(m + n)$, we may write the fraction as

$$\frac{(m + n)(m^2 - mn + n^2)}{(m - n)(m + n)} = \frac{m^2 - mn + n^2}{m - n}.$$

Since in our case $m - n = 1$, the value of this expression is

$$\begin{aligned} 2016^2 - 2016 \times 2015 + 2015^2 &= 2016(2016 - 2015) + 2015^2 \\ &= 2016 + 2015^2 \\ &= 4\,062\,241. \end{aligned}$$

Alternative

(This solution does not use the result of part (a).)

Write $m = 2016$, $n = 2015$; then $m = n + 1$. Using the binomial expansion, we get

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

Therefore the fraction is

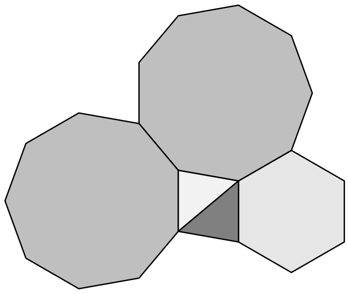
$$\frac{(n^3 + 3n^2 + 3n + 1) + n^3}{(n^2 + 2n + 1) - n^2} = \frac{2n^3 + 3n^2 + 3n + 1}{2n + 1}.$$

Using algebraic division, we obtain

$$n^2 + n + 1 = 2015^2 + 2015 + 1 = 4\,062\,241.$$

2. The diagram shows five polygons placed together edge-to-edge: two triangles, a regular hexagon and two regular nonagons.

Prove that each of the triangles is isosceles.



Solution

Commentary

A triangle is isosceles if it has two sides of equal length. There are several regular polygons in this question, so it seems reasonable to start by identifying all lines of equal length.

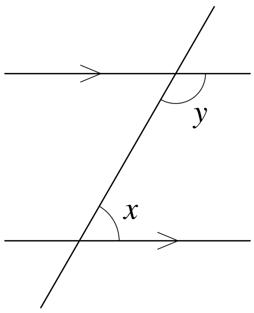
If this doesn't complete the proof we can start looking at angles. If a triangle has two equal angles then it is isosceles.

We will almost certainly need to work out angles in a regular hexagon and a regular nonagon. We can do this in various ways. For example, we can find the sum of angles in a nonagon by splitting it into seven triangles (by drawing all the diagonals from one vertex). Alternatively, we can use the fact that the exterior angles of any polygon add up to $360°$; this means that in a regular n-sided polygon each exterior angle is $360° \div n$ so each interior angle is $180° - 360° \div n$. We could also draw isosceles triangles with one vertex at the centre of the regular polygon. Either way, we find that each interior angle of a regular hexagon is $120°$ and each interior angle of a regular nonagon is $140°$.

It is a good idea to label some points on the diagram so you can refer to various triangles and sides.

Our alternative solution uses properties of parallel lines and parallelograms. More precisely, it uses *converses* of two results about parallel lines and parallelograms.

The first result is about angles on parallel lines. If two parallel lines are intersected by a third line, then the interior angles (marked x and y in the diagram) add up to $180°$.

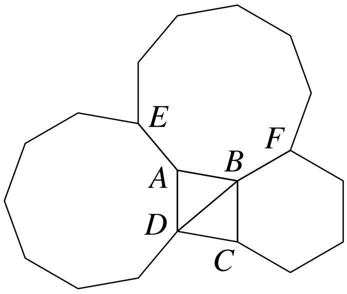


The converse of this result states that, if two lines are intersected by a third line so that angles x and y add up to 180°, then the two lines are parallel.

The second result is about sides of a parallelogram. A parallelogram is defined as a quadrilateral with two pairs of parallel sides. One property of a parallelogram is that opposite sides are equal in length. One possible converse of this result is that, if a quadrilateral has one pair of equal and parallel sides, then it is a parallelogram.

You should note that converse statements aren't always true. The two converse statements above are true and here you can use them without proof, although it is a good exercise to try to prove them.

Label some of the vertices, as shown in the diagram.



First we show that triangle ABD is isosceles.

Since the nonagons and the hexagon are regular and meet edge-to-edge,, we have

$$DA = AE = AB.$$

This means that triangle ABD is isosceles.

We now look at triangle BCD. From the regular polygons, we find that

$$BC = BF = AB,$$

but we don't know anything about sides CD and BD. So let's calculate some angles.

Each angle in a regular hexagon is 120° and each angle in a regular nonagon is 140°. Therefore,

$$\begin{aligned}\angle DAB &= 360° - 140° - 140° \\ &= 80°\end{aligned}$$

and

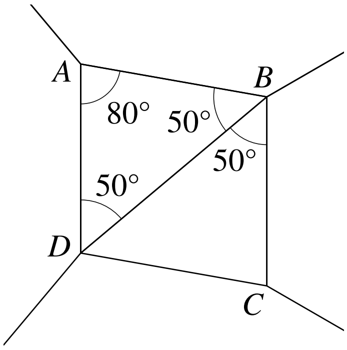
$$\begin{aligned}\angle ABC &= 360° - 140° - 120° \\ &= 100°.\end{aligned}$$

Since we already know that the triangle ABD is isosceles, we can calculate

$$\begin{aligned}\angle ABD &= \frac{180° - 80°}{2} \\ &= 50°\end{aligned}$$

and therefore

$$\begin{aligned}\angle DBC &= 100° - 50° \\ &= 50°.\end{aligned}$$



We cannot directly find any other angles in triangle BCD. However, triangle BCD is congruent to triangle BAD because

$$\begin{aligned} BC &= AB, && \text{(from regular polygons)} \\ BD &= BD && \text{(common side)} \\ \text{and } \angle CBD &= \angle ABC. && \text{(each equals } 50^\circ\text{)} \end{aligned}$$

Since triangle BAD is isosceles, so is triangle BCD (with $BC = CD$).

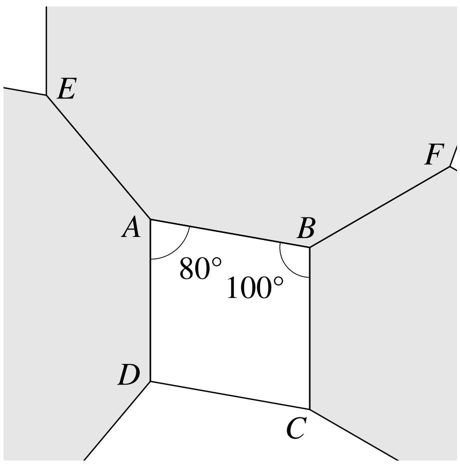
Alternative

Each angle in a regular hexagon is 120° and each angle in a regular nonagon is 140°. Therefore,

$$\begin{aligned} \angle DAB &= 360^\circ - 140^\circ - 140^\circ \\ &= 80^\circ \end{aligned}$$

and

$$\begin{aligned} \angle ABC &= 360^\circ - 140^\circ - 120^\circ \\ &= 100^\circ. \end{aligned}$$



These two angles add up to 180° so, by the converse of internal angles on parallel lines, we find that AD and BC are parallel.

Since the nonagons and the hexagon are regular and meet edge-to-edge, we have

$$AD = EA = AB = BF = BC.$$

Hence AD and BC are equal and parallel. It follows that the quadrilateral $ABCD$ is a parallelogram. Therefore the sides AB and CD are equal.

But we already know that $AB = BC = DA$, so all four sides of $ABCD$ are in fact equal. Therefore the two required triangles are isosceles.

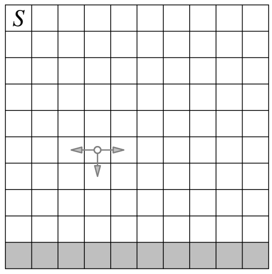
NOTE

The two triangles are in fact congruent, and the quadrilateral $ABCD$ is a rhombus.

3. A ladybird is going for a wander around a 10×10 board, subject to the following three rules (see the diagram).

(i) She starts in the top left cell, labelled S.
(ii) She only moves left, right or down, as indicated.
(iii) She never goes back to a cell that she has already visited.

In how many different ways can she reach the bottom row of cells, shaded grey?



Solution

Commentary

One possible approach would be to try to systematically list all possible paths around the board. From the starting square the ladybird can either move down or right. If she goes down she then again has a choice of down or right; if she goes right she can continue to the right or go down, but cannot go left along the same level (because she is not allowed to return to a cell she has already visited).

While this method is certainly systematic, there are too many options to list. So let's stop and think what the rules really mean for the ladybird's route around the board.

At any stage, the ladybird can move down a row. In order to reach the bottom, she will need to move down a row exactly nine times (remember that she can't move up).

Within a row, she can move down from any of the ten cells. Rule (iii) says that she never goes back to a cell that she has already visited. This means that when she arrives at a row, she must move to the left or right or stay still, and then go down. She can't go left for a bit then right for a bit, for example, or she'd go back on herself.

This is a key observation. Imagine recording an accurate description of the ladybird's route through the chessboard. In principle, it might be a long sequence of steps to the left, steps to the right, and steps down. But we have seen that the rules mean that it is simpler than this. She takes a total of 9 steps down, and in between each step she navigates directly to the cell from which she'll make the next step down. So to record her route precisely, it's enough to write a list of the nine cells from which she takes a step down — there is no ambiguity about her route in between those cells, she must just step down and then move directly to the next cell in the list. This gives us a much more manageable counting problem.

Once the ladybird (here called Ella) has reached the dth row she can move left or right, or not at all, to one of 10 cells in that row before her descent to the next row (unless she has already reached the last row).

Since Ella can't go back on herself, there is only one way for her to reach any of the ten cells in dth row.

For each of rows 1, 2, ..., 9, Ella has 10 options for the point at which she descends. Therefore there are 10^9, that is, 1 000 000 000, ways for Ella to reach the bottom row of cells.

NOTE

Wherever Ella starts in the top row, the answer is the same.

Also, if the board is of shape $m \times n$ (that is, m rows, n columns) then Ella can reach the bottom row from anywhere in the top row in n^{m-1} different ways.

4. (a) A tournament has n contestants. Each contestant plays exactly one game against every other contestant. Explain why the total number of games is $\frac{1}{2}n(n-1)$.

(b) In a particular chess tournament, every contestant is supposed to play exactly one game against every other contestant. However, contestant A withdrew from the tournament after playing only ten games, and contestant B withdrew after just one game.

A total of 55 games were played.

Did A and B play each other?

Solution

(a)

Commentary

There are quite a lot of ways of counting the number of games! We give some examples.

We could consider each contestant in turn. The first contestant plays $n-1$ games (against every contestant other than themselves). For the second contestant we have already counted the game against the first contestant, so she plays $n-2$ other games. Continuing this reasoning, we find that the total number of games is

$$(n-1)+(n-2)+\cdots+2+1.$$

There are a good few ways of working out this sum.

One way is to compare the sum to its reverse. If we call the sum S, then we have both

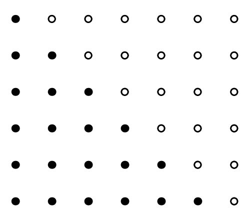
$$\begin{aligned} (n-1)+(n-2)+\cdots+2+1 &= S, \\ \text{and } 1+2+\cdots+(n-2)+(n-1) &= S. \end{aligned}$$

By adding corresponding terms, we get

$$(n-1+1)+(n-2+2)+\cdots+(2+n-2)+(1+n-1)=2S.$$

The left-hand side of this equation is a sum of $n-1$ bracketed terms, each of which is equal to n, so the sum is equal to $n(n-1)$. Thus $S=\frac{1}{2}n(n-1)$.

Yet another way is more geometrical. The sum can be represented by a triangle of dots: one dot in the first row, two dots in a second, and so on. The diagram shows two such triangle fitted together to form a rectangle.



> If there are $n-1$ rows of dots, the rectangle has $n-1$ rows and n columns, and hence contains $n(n-1)$ dots in total. But the original sum represents half of this rectangle, so the result follows.
>
> On the other hand, you may know about arithmetic series in general. This is an arithmetic series with $n-1$ terms, first term 1 and common difference 1. Therefore the sum is
>
> $$\frac{n-1}{2}[2+(n-2)] = \frac{1}{2}n(n-1).$$
>
> Finally, we can go about it differently. Instead of forming a sum, we can look at the number of ways to pair the contestants. We need to be careful not to count pairs twice. This is the solution we present below.

Contestants never play themselves but do play all the other $n-1$ competitors. Therefore there are $n(n-1)$ pairs (X, Y) such that X plays Y, with the contestants named in that order.

The game in which X plays Y is the same as that in which Y plays X however, so naming the contestants in order means that each game is counted twice. Therefore the total number of games is $\frac{1}{2}n(n-1)$.

(b)

> Commentary
>
> We need to check whether it is possible to have a total of 55 games in the two cases: when A and B played each other, and when they didn't.
>
> One way to do this would be to count the number of games that don't involve A or B and then to add in the number of games played by A and by B. Another way would be to count the total number of games if A and B had played the whole tournament, and then to subtract the numbers of games missed by A and B because they withdrew early.
>
> In each case, we know that the total number of games played is 55. We don't know how many contestants there were at the start, so give this number a name (say m). We'll get an equation involving m that we can then try to solve. We'll also need to be careful to remember that A and B were scheduled to play each other, so we mustn't count that game twice.

Suppose that there are m contestants other than A and B. The number of games those m contestants played with each other is $\frac{1}{2}m(m-1)$.

There are an additional 10 games played by A. If those include a game between A and B, then there are no additional games played by B, so the total number of games played is

$$\frac{1}{2}m(m-1) + 10 = 55.$$

If B did not play A then there is one additional game played by B, so the total number of games is

$$\frac{1}{2}m(m-1) + 10 + 1 = 55.$$

In the first case, $m(m-1) = 90$, which is possible when $m = 10$. In the second case, $m(m-1) = 88$. The last equation has no integer solutions, since $m(m-1) \leq 72$ when $m \leq 9$ and $m(m-1) \geq 90$ when $m \geq 10$.

Thus A and B *did* play each other.

Note

There were 12 contestants at the start of the tournament.

5. (a) The integer N is a square. Find, with proof, all possible remainders when N is divided by 16.

(b) Find all positive integers m and n such that

$$m! + 76 = n^2.$$

[*The notation* $m!$ *stands for the* factorial *of* m, *that is,* $m! = m\times(m-1)\times\cdots\times 2\times 1$. *For example,* $4! = 4\times 3\times 2\times 1$.]

SOLUTION

(a)

COMMENTARY

It seems reasonable to start by working out remainders of some square numbers; you may soon notice a pattern. However, we need to produce a general argument.

If number n gives remainder r when divided by 16, we can write it as $n = 16k + r$. The possible values of r are $0, 1, 2, \ldots, 15$. For example, if n gives remainder 5 when divided by 16 then $n = 16k + 5$, so

$$n^2 = (16k + 5)^2 = 256k^2 + 160k + 25.$$

Notice that the first two terms are divisible by 16, so n^2 gives the same remainder as 25 when divided by 16.

We can repeat this calculation for each r from 0 to 15 to get the list of all possible remainders. It is worth noticing a slight shortcut: the remainders seem to repeat from $r = 8$. The solution below demonstrates why this is the case.

Write $N = n^2$ and $n = 8k + r$ where r, the remainder when n is divided by 8, is one of 0, 1, 2, ..., 7. Then $N = (8k + r)^2 = 64k^2 + 16kr + r^2$, and this leaves the same remainder as does r^2 when divided by 16.

Those remainders R are shown in the following table.

r	R
0	0
1	1
2	4
3	9
4	0
5	9
6	4
7	1

Thus the possible remainders when N is divided by 16 are 0, 1, 4 and 9.

(b)

> COMMENTARY
>
> Part (a) suggests that it may be helpful to look at remainders when dividing by 16. The possible remainders of n^2 are 0, 1, 4 and 9, and 76 gives remainder 12. So $m!$ needs to give the remainder 4, 5, 8 or 13 when divided by 16.
>
> Looking at some factorial numbers suggests that after a certain point, they are all divisible by 16. This is because 6! is a multiple of 16 and all further factorial numbers are multiples of 6! (e.g., $7! = 6! \times 7$, $8! = 6! \times 7 \times 8$ and so on).
>
> This means that m cannot be greater or equal to 6. We only need to check positive integers smaller than 6 to find all the solutions.

If $m \geq 6$ then $m!$ is a multiple of $6! = 720$, which itself is a multiple of 16. Therefore the remainder when $m! + 76$ is divided by 16 is the same as the remainder when 76 is divided by 16, which is 12. From part (a) we know, then, that $m! + 76$ cannot be a square if $m \geq 6$.

If m is 1, 2 or 3, then the remainders after division of $m! + 76$ by 16 are 13, 14 and 2 respectively, and so $m! + 76$ cannot be a square.

But if $m = 4$ then $m! + 76 = 24 + 76 = 100 = 10^2$, and if $m = 5$ then $m! + 76 = 120 + 76 = 196 = 14^2$. Therefore the only pairs (m, n) of positive integers such that $m! + 76 = n^2$ are $(4, 10)$ and $(5, 14)$.

NOTE

We could look at remainders on division by some other numbers. For example, a similar argument works if we look at division by 7. For $m \geq 7$, $m! + 76$ gives remainder 6 when divided by 7, and we can show that a square number cannot give this remainder.