Mathematical Olympiad for Girls

Tuesday 29th September 2015

Organised by the United Kingdom Mathematics Trust

Instructions

1. Do not turn over until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks. Full marks will be awarded for written solutions — not just answers — with complete proofs of any assertions you may make.

   Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt.
4. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.

   However, one complete solution will gain more credit than several unfinished attempts.
5. Earlier questions tend to be easier. Some questions have two parts. Part (a) introduces results or ideas useful in solving part (b).
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Start each question on a fresh sheet of paper. Write on one side of the paper only.

   On each sheet of working write the number of the question in the top left-hand corner and your name, initials and school in the top right-hand corner.
8. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
9. Staple all the pages neatly together in the top left hand corner.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Wednesday 30th September.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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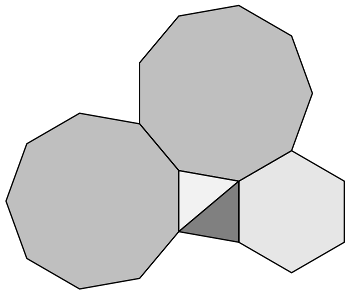
1. (a) Expand and simplify $(a - b)(a^2 + ab + b^2)$.

(b) Find the value of

$$\frac{2016^3 + 2015^3}{2016^2 - 2015^2}.$$

2. The diagram shows five polygons placed together edge-to-edge: two triangles, a regular hexagon and two regular nonagons.

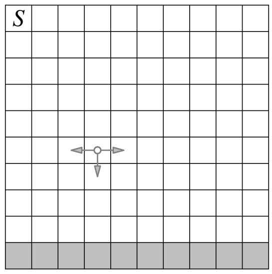
Prove that each of the triangles is isosceles.



3. A ladybird is going for a wander around a 10×10 board, subject to the following three rules (see the diagram).

(i) She starts in the top left cell, labelled S.
(ii) She only moves left, right or down, as indicated.
(iii) She never goes back to a cell that she has already visited.

In how many different ways can she reach the bottom row of cells, shaded grey?



4. (a) A tournament has n contestants. Each contestant plays exactly one game against every other contestant. Explain why the total number of games is $\frac{1}{2}n(n - 1)$.

(b) In a particular chess tournament, every contestant is supposed to play exactly one game against every other contestant. However, contestant A withdrew from the tournament after playing only ten games, and contestant B withdrew after just one game.

A total of 55 games were played.

Did A and B play each other?

5. (a) The integer N is a square. Find, with proof, all possible remainders when N is divided by 16.

(b) Find all positive integers m and n such that

$$m! + 76 = n^2.$$

[*The notation* $m!$ *stands for the* factorial *of* m, *that is,* $m! = m \times (m - 1) \times \cdots \times 2 \times 1$. *For example,* $4! = 4 \times 3 \times 2 \times 1$.]

