Mathematical Olympiad for Girls

Organised by the United Kingdom Mathematics Trust

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

The mark allocation on Mathematics Olympiad papers is different from what you are used to at school. To get any marks, you need to make significant progress towards the solution. This is why the rubric encourages candidates to try to finish whole questions rather than attempting lots of disconnected parts.

Each question is marked out of 10.

3 or 4 marks roughly means that you had most of the relevant ideas, but were not able to link them into a coherent proof.

8 or 9 marks means that you have solved the problem, but have made a minor calculation error or have not explained your reasoning clearly enough. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap?

Enquiries about the Mathematical Olympiad for Girls should be sent to:

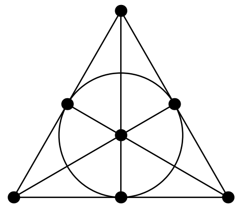
MOG, UK Mathematics Trust, School of Mathematics Satellite, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. The diagram shows a figure consisting of six line segments and a circle, each containing three points.

Each point is labelled with a real number. The sum of the three numbers on each line segment or circle is T.

Prove that each number is equal to $\frac{1}{3}T$.



Solution

Commentary

This question is about sums of numbers, so a sensible first step seems to be to give each number a name so we can write some equations.

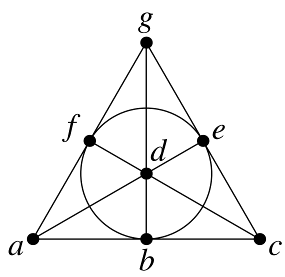
Label the numbers as shown in the diagram below. There are seven unknowns and we can write seven equations. You may know several different methods for solving such systems of equations, for example elimination and substitution. In this case, substituting from one equation into another is likely to produce long expressions.

A better method is to look for equations which share one or more unknowns and eliminate those unknowns. For example, $a + b + c = T$ and $a + d + e = T$. You can subtract the two equations to obtain $b + c - d - e = 0$ and so $b + c = d + e$.

You can then look for another two equations that contain those four unknowns, for example $g + d + b = T$ and $g + e + c = T$. Subtracting those two gives $d + b = e + c$. Adding this to the equation we found above gives $2b + c + d = 2e + c + d$ and so $b = e$.

You can use the symmetry of the situation to see that we can produce an analogous proof showing that $b = f$. Having found that several of the unknowns are equal to each other, it may be a good idea to relabel the diagram to show this. We can then complete the proof, as we have done below.

Let the numbers labelling the points be as shown in the diagram.

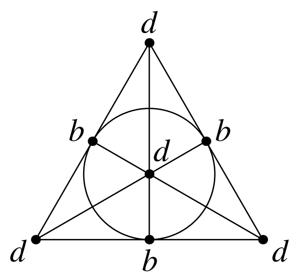


Since $T = a + b + c = a + d + e$, we have $b + c = d + e$. Also, $T = b + d + g = c + e + g$ and so $b + d = c + e$. Therefore $2b + c + d = 2e + c + d$ and so $b = e$.

Substituting this back into $b + d = c + e$ we obtain $d = c$.

Analogously, we can prove that $b = f$, and that $d = a$ and $d = g$.

We can therefore relabel the diagram as shown on the right.



Considering the three points on the circle, we have $3b = T$ and so $b = \frac{1}{3}T$. Considering the base of the triangle, $2d + b = T$ and so $d = \frac{1}{3}T$ as well.

Thus each number is equal to $\frac{1}{3}T$, as required.

Note

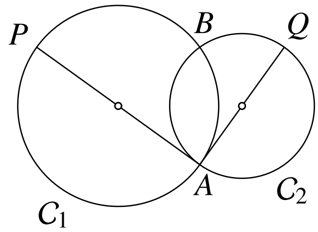
The configuration in the question is known as the *Fano plane* (after Gino Fano, 1871–1952).

The Fano plane is an example of a "finite projective plane". It has only seven points and seven lines (represented as the six line segments and the circle in our diagram). Notice that every pair of lines have one common point; hence, in this plane, every two lines intersect and there are no parallel lines.

2. The diagram shows two circles C_1 and C_2 with diameters PA and AQ. The circles meet at the points A and B, and the line PA is a tangent to C_2 at A.

Prove that

$$\frac{PB}{BQ} = \frac{\text{area}\,C_1}{\text{area}\,C_2}.$$



Solution

Commentary

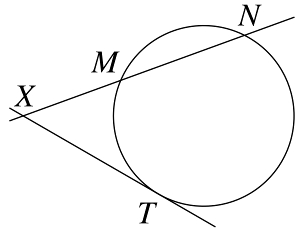
In geometry questions, there are often lots of different extra lines we could draw, and different lengths and angles we could calculate, so it can be difficult to decide where to start. A useful strategy is to think about what we are trying to prove, and focus on lines, angles or triangles which we think might be useful. This is essentially "working backwards" from the answer; in writing up the solution, you need to be careful to start with the given facts and end with the required conclusion.

Results about ratios of lengths can often be proved using similar triangles. In this question we are also interested in the ratio of areas of the two circles. But the area of the circle is proportional to the square of its diameter, so the required result is equivalent to $\dfrac{PB}{BQ} = \dfrac{AP^2}{AQ^2}$.

The four lengths from the above equation appear in triangles ABP and ABQ. If you can show that those two triangles are similar, you may be able to use the ratios of their sides to get the required result.

To prove that two triangles are similar you need to find two pairs of equal angles. At the first glance it looks like there are no angles given in this question. However, diameters and tangents in circles create right angles, so this is a good place to start.

We present three possible solutions. The first two use similar triangles. The third uses the tangent-secant theorem, which says that in the diagram on the right, if XT is a tangent to the circle, then $XM \times XN = XT^2$.

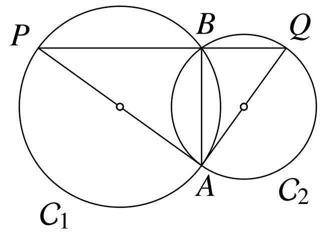


The third solution also uses the result that the angle between a diameter and the tangent at an endpoint of the diameter is a right angle. In fact it uses this result twice. First we use the fact that PA is tangent to C_2 to conclude that PAQ is a right angle. Then we use the *converse* of the result: since PAQ is a right angle and AP is a diameter of C_1, AQ is also a tangent to C_1.

Method 1

The angle in a semicircle is a right angle. Since AP is a diameter of C_1, $\angle PBA = 90°$, and since AQ is a diameter of C_2, $\angle ABQ = 90°$.

Also, the angle between the tangent and the diameter of a circle at the point of contact is a right angle. Since PA is a tangent to C_2, $\angle PAQ = 90°$. Hence $\angle APB = 90° - \angle PAB = \angle QAB$.



Each of triangles PBA and ABQ has a right angle at B, and $\angle APB = \angle QAB$. Hence they are similar (AA). Therefore, by considering the ratios of their sides, we obtain

$$\frac{PB}{AB} = \frac{PA}{AQ}$$

so that

$$PB = \frac{PA \times AB}{AQ}, \tag{1}$$

and again from the similar triangles

$$\frac{BA}{BQ} = \frac{PA}{AQ}$$

so that

$$BQ = \frac{BA \times AQ}{PA}. \tag{2}$$

Dividing equation (1) by equation (2), we get

$$\frac{PB}{BQ} = \frac{PA^2}{AQ^2}.$$

But area $C_1 = \frac{1}{4}\pi AP^2$ and area $C_2 = \frac{1}{4}\pi AQ^2$. Therefore $\dfrac{PB}{BQ} = \dfrac{\text{area } C_1}{\text{area } C_2}$, as required.

Method 2

As in Method 1, $\angle PBA = 90°$ and $\angle ABQ = 90°$. Therefore

$$\angle PBQ = \angle PBA + \angle ABQ = 180°,$$

and so PBQ is a straight line segment.

Triangles PAQ and PBA share an angle at P and have right angles at A and B respectively, so they are similar (AA).

Triangles PAQ and ABQ also share an angle, the one at Q, and also have right angles at A and B respectively. Therefore they too are similar (AA).

From the similarity $PBA \sim PAQ$ we have $\dfrac{PB}{PA} = \dfrac{PA}{PQ}$, and from the similarity $ABQ \sim PAQ$ we have $\dfrac{BQ}{AQ} = \dfrac{AQ}{PQ}$. Therefore

$$PB = \frac{PA^2}{PQ} \quad \text{and} \quad BQ = \frac{AQ^2}{PQ},$$

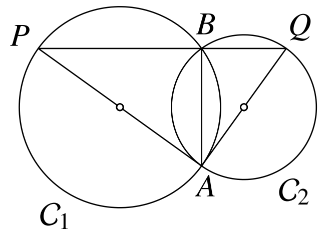
so that

$$\frac{PB}{BQ} = \frac{PA^2}{AQ^2},$$

which is equivalent to the required result.

Method 3

First, prove that PBQ is a straight line segment as in the previous solution.



Since PA is tangent to the circle C_2 at A, $\angle PAQ = 90°$. But AP is the diameter of C_1, so AQ is a tangent to C_1 at A.

Using the tangent-secant theorem for circle C_2 and point P, we obtain

$$PB \times PQ = PA^2.$$

Using the same theorem for circle C_1 and point Q, we get

$$QB \times QP = QA^2.$$

Dividing the first equation by the second gives $\frac{PB}{QB} = \frac{PA^2}{QA^2}$, and the required result follows.

3. Punam puts counters onto some of the cells of a 5×5 board. She can put more than one counter on each cell, and she can leave some cells empty. She tells Quinn how many counters there are in each row and column. These ten numbers are all different.

Can Quinn always work out which cells, if any, are empty?

SOLUTION

COMMENTARY

It is usually a good idea to start by trying to produce some examples of arrangements of counters that satisfy the given condition.

You then need to decide what you are trying to prove, drawing on your experience from experimenting with some examples of ways of arranging the counters.

If you think that Quinn can always work out which cells are empty then you need to show that this is the case for all possible arrangements for which the ten row and column totals are all different.

If you think that Quinn cannot identify the empty cells, then you need to find an example of two different arrangements which have the same row and column totals, but the empty cells in different places.

No, it is not always possible for Quinn to identify the empty cells. Consider two arrangements X, Y of Punam's counters that are the same except in the top left 2×2 square, where, say, X has

0	1
1	0

while Y has

1	0
0	1

.

For example, X and Y could be as shown in the figures below.

0	1	0	0	0
1	0	1	0	0
0	0	0	0	3
0	3	1	0	0
9	8	7	6	5

X

1	0	0	0	0
0	1	1	0	0
0	0	0	0	3
0	3	1	0	0
9	8	7	6	5

Y

The row totals in these examples are 1, 2, 3, 4, 35 and the column totals are 10, 12, 9, 6, 8, all different.

Since Quinn knows only the total numbers of counters in each row and column, she could not distinguish X from Y, and therefore she cannot work out which cells are empty.

NOTE

If the problem is modified so that Punam can put at most one counter on each cell, with the condition that all the row totals are different and all the column totals are different, then Quinn *can* use the row and column totals to work out which cells are empty. Can you prove this?

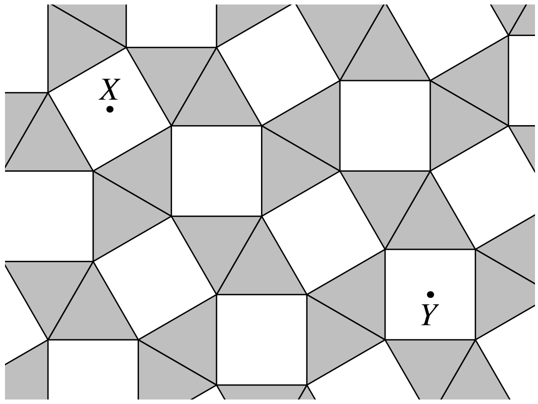
4. (a) In the trapezium $ABCD$, the edges AB and DC are parallel. The point M is the midpoint of BC, and N is the midpoint of DA.

Prove that $2MN = AB + CD$.

(b) The diagram shows part of a tiling of the plane by squares and equilateral triangles.

Each tile has edges of length 2. The points X and Y are at the centres of square tiles.

What is the distance XY?



Solution

(a)

Commentary

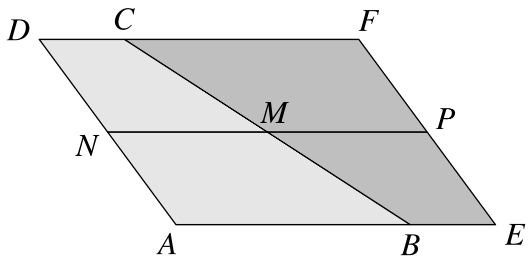
There are several different ways to approach this part, and we present two possible proofs here.

In the first proof, since we are interested in the length $AB + CD$, we are going to extend the two bases of the trapezium, as shown in the diagram below. The two identical copies of the trapezium make up a parallelogram with base length $AB + CD$, so we just need to prove that NMP is a straight line parallel to the base.

You may have seen this construction when deriving the formula for the area of the trapezium. Our second proof explicitly uses the area of the trapezium: we create a rectangle with base length NM and the area equal to the area of the trapezium.

Method 1

Extend the side AB to point E and the side DC to point F such that $BE = DC$ and $CF = AB$. Then $AEFD$ is a parallelogram, since the opposite sides AE and DF are parallel and both have length $AB + CD$. It follows that EF and AD are parallel and equal in length.



Let P be the midpoint of EF. Then AN and EP are parallel and have equal lengths, so $AEPN$ is also a parallelogram, and hence $NP = AE = AB + CD$.

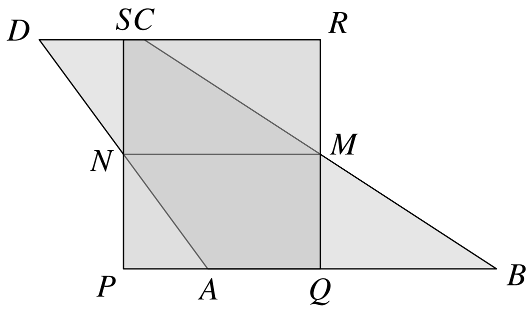
The trapezia $ABCD$ and $FCBE$ are congruent: we have already proved that the corresponding sides are equal, $\angle BAD = \angle CFE$ and $\angle ADC = \angle FEB$ from the parallelogram, and $\angle ABC = \angle FCB$ because the lines AE and DF are parallel.

M is the midpoint of BC, which is the shared side of the two trapezia.Therefore $\angle NMC = \angle BMP$. It follows that NMP is a straight line. The congruence of the two trapezia also implies that $NM = MP$.

Therefore $NP = 2MN$ and so $2MN = AB + CD$, as required.

Method 2

Draw a line through N perpendicular to AB, and let it meet AB at P and DC at S. Draw another line perpendicular to AB through M, and let it meet AB at Q and DC at R, as shown in the diagram. Note that some of the P, Q, R and S will be on the sides and some on the extensions of sides AB and DC; it can be checked that the proof works in all possible cases.



$PQRS$ has four right angles, so it is a rectangle. Its base is equal in length to MN and its height is equal to the height of the trapezium, h. Hence the area of the rectangle $PQRS$ is $MN \times h$.

Triangles PAN and SDN are congruent: they are both right-angled, have equal angles at N and AN=DN (since N is the midpoint of AD). Similarly, triangles QBM and RCM are congruent. Hence the area of the rectangle $PQRS$ equals the area of the trapezium $ABCD$. Therefore we have:

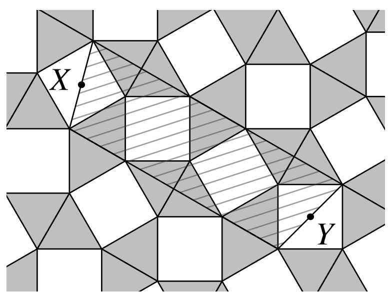
$$MN \times h = \tfrac{1}{2}(AB + CD)h.$$

It follows that $2MN = AB + CD$, as required.

(b)

Commentary

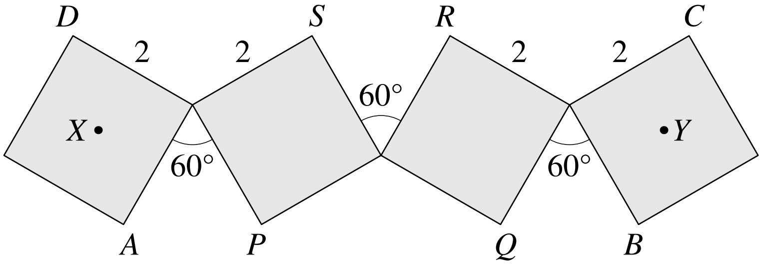
The first thing you should ask is how you can use the result from part (a). X and Y are midpoints of the diagonals of the two squares, so it seems sensible to look for a trapezium with those two diagonals as sides. The parallel sides of this trapezium are made up of the sides and heights of the equilateral triangle, so you can calculate their lengths.



There are two things you need to prove before you can do the calculations. First, you need to show that the sides labeled DC and AB in the diagram below are in fact parallel. Second, you need to show that those lines pass through the points S, R, P and Q.

We claim that $XY = 3 + 3\sqrt{3}$. Here is a proof.

Extract four squares from the shaded part of the pattern, as shown in the diagram below. Let A, B, C, D be the vertices of the squares containing the points X, Y, as shown. Also, going from left to right on the "lower" zig-zag boundary of the figure, label the "bottom" vertices of the two inner squares P, Q, and going from right to left on its "upper" boundary, label the two "top" vertices of those squares R, S.



The bisector of the middle 60° angle is a line of symmetry of the figure. Reflection in that line interchanges A and B and interchanges C and D. Consequently both AB and CD are perpendicular to that line, and so AB is parallel to CD, that is, $ABCD$ is a trapezium. Therefore, using the result of part (a), $2XY = AB + CD$.

The isosceles triangle with base AP has angle 60° at its apex, so its other two angles are 60° also, and therefore it is equilateral. The isosceles triangle that has PQ as its base has angle 120° at its apex, hence angles 30° at P and Q.

Consequently $\angle APQ = 60° + 90° + 30° = 180°$, that is, APQ is a straight line. Similarly (or by symmetry) PQB is a straight line. Thus the line segment AB passes through P and Q. A very similar argument shows that the line segment CD passes through R and S.

Now $AP = QB = SR = 2$, since those are the sides of equilateral triangles.

The length PQ is equal to twice the height of the equilateral triangle, as can be seen from the diagram of the full tiling above (in the commentary). Using Pythagoras' Theorem, the height of the equilateral triangle of side 2 is $\sqrt{3}$. Hence $PQ = DS = RC = 2\sqrt{3}$.

Therefore $AB = 2 + x + 2 = 4 + 2\sqrt{3}$ and $DC = x + 2 + x = 2 + 4\sqrt{3}$. By part (a), $2XY = (AB + CD) = 6 + 6\sqrt{3}$, so $XY = 3 + 3\sqrt{3}$ as claimed.

5. Alia, Bella and Catherine are multiplying fractions, aiming to obtain integers. Each of them can multiply as many fractions as she likes (including just one), and can use the same fraction more than once.

Alia's fractions are of the form $\frac{n+1}{n}$, where n is a positive integer.

Bella's fractions are of the form $\frac{6p-5}{3p+6}$, where p is a positive integer.

Catherine's fractions are of the form $\frac{4q-1}{2q+1}$, where q is a positive integer.

Which integers can each of them obtain?

Solution

Commentary

The first thing to do is to try multiplying some fractions and see what integers you can get. Hopefully you can find how to obtain any integer greater than 1 using Alia's fractions, and decide that Bella cannot obtain any integers.

Catherine's task is more challenging. You may want to start by listing several of her fractions — $\frac{3}{3}, \frac{7}{5}, \frac{11}{7}, \frac{15}{9}, \ldots$ — and seeing what integers can be obtained from them. After some experimenting, you may start to suspect that you can obtain larger odd integers by using some of the smaller ones. For example, if you can obtain 7 then you can use it to obtain 11 by doing $\frac{11}{7} \times 7$.

Integers that do not appear as numerators of Catherine's fractions are a bit more difficult. For example, to obtain 13 you need to realise that 39 appears as a numerator in $\frac{39}{21}$, and so you can obtain 13 as $\frac{39}{21} \times 7$.

This suggests that you should look for a slightly different calculation depending on whether the required odd integer is of the form $4m - 1$ or $4m + 1$. Integers of the form $4m - 1$ appear as numerators of Catherine's fractions, so if you can obtain $2m + 1$ then you can also obtain $4m - 1 = (2m + 1) \times \frac{4m-1}{2m+1}$. To obtain an integer of the form $4m + 1$ you need to look at $3(4m + 1) = 12m + 3$, because this does appear as the numerator of the fraction $\frac{12m+3}{6m+3}$.

Taking $n = 1$, Alia gets the integer 2. Now if $N > 2$, then Alia can obtain N as $\frac{2}{1} \times \frac{3}{2} \times \cdots \times \frac{N}{N-1}$. Thus Alia can obtain any positive integer except 1. (She cannot obtain 1 because all her fractions are greater than 1.)

Bella, however, cannot obtain any positive integers. Her store consists of fractions of the form

$\frac{N}{3D}$, where the positive integer N is not a multiple of 3.

If she multiplies m of these fractions together she will obtain a fraction of the form $\frac{X}{3^m Y}$, where the positive integer X is not multiple of 3. Since the term 3^m cannot cancel, this will never be a positive integer (we must have $m \geq 1$).

Since the fractions available to Catherine have odd numerator and odd denominator, the same will be true of any fraction she can create by multiplying them. Therefore she certainly cannot reach any even positive integers. She can, however, reach every odd positive integer.

This may be seen as follows. Let $f(q)$ be the fraction $\frac{4q-1}{2q+1}$.

Then $f(1) = 1$, $f(4) \times f(7) = \frac{15}{9} \times \frac{27}{15} = 3$, and $3 \times f(4) = 3 \times \frac{15}{9} = 5$. Thus Catherine can obtain 1, 3 and 5.

Suppose now that $m \geq 2$ and Catherine has managed to obtain all odd numbers up to and including $4m - 3$. We know that she can do this for $m = 2$. Since $2m + 1 \leq 4m - 3$, she can obtain $2m + 1$, and so she can use the calculations

$$4m - 1 = (2m + 1) \times \frac{4m - 1}{2m + 1} = (2m + 1) \times f(m) \quad \text{and}$$
$$4m + 1 = (2m + 1) \times \frac{12m + 3}{6m + 3} = (2m + 1) \times f(3m + 1)$$

to add $4m - 1$ and $4m + 1$ to her list, so as to extend it to all odd numbers up to and including $4m + 1 = 4(m + 1) - 3$.

This means that, having obtained 5, she can obtain 7 and 9, then 11 and 13, and so on. Hence Catherine can obtain any odd positive integer.

Note

The method we used for Catherine, where we prove a result about a certain positive integer by using the same result for a smaller integer, is called *proof by induction*. You may learn about it in your future studies; it is a very useful method for proving results about positive integers.