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Mathematical Olympiad for Girls 2017

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has some sort of overall strategy or not. An answer which is essentially a solution, but might contain either errors of calculation, flaws in logic, omission of cases or technical faults, will be marked on a '10 minus' basis. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

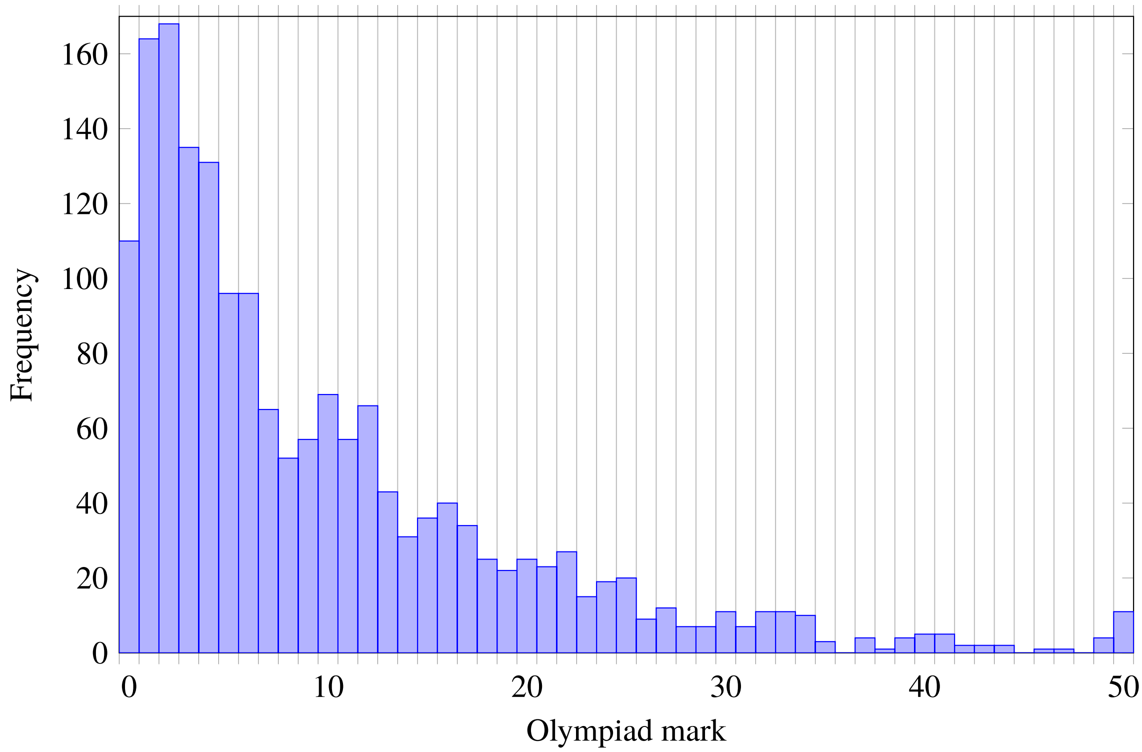
The markers were impressed with the way candidates engaged with this paper, with many scoring marks on at least two different questions. Most understood what the questions were asking and were at least able to produce some useful examples. Particularly good solutions were seen on Question 4, where a majority of candidates adopted a sensible algebraic approach.

There was a large number of excellent solutions and clear explanations. It was pleasing to see so many candidates have a good understanding that a proof needs to cover all possibilities rather than just special examples. In particular, in Question 3, many candidates realised that they needed to look at several different cases for the configuration of the four points, even if they were not able to complete the argument for all of them.

Although listing lots of examples does not in general constitute a proof, trying some examples if often a useful way to understand a question and get ideas for how to proceed. For instance, in Question 1, constructing some examples might lead to the idea of a substitution which turns a configuration with total T into one with total $40 - T$. If such examples turn out to be useful they may be awarded marks, so it is a good idea to write them down.

The 2017 Mathematical Olympiad for Girls attracted 1756 entries. The scripts were marked on 14th and 15th October in Cambridge by a team of Abdul Hadi Khan, Andrea Chlebikova, Andrew Carlotti, Eszter Backhausz, Eve Pound, Georgina Majury, Ina Hughes, Jack St Clair, James Cranch, James Harris, Jasmina Lazic, Jerome Watson, John Haslegrave, Joseph Myers, Kasia Warburton, Lawrence Hollom, Maria Holdcroft, Martin Orr, Mary Teresa Fyfe, Peter Neumann, Philip Coggins, Renzhi Zhou, Richard Freeland, Robin Bhattacharyya, Sam Maltby, Sylvia Neumann, Tim Cross, Tom Bowler, Vesna Kadelburg and Warren Li.

Mark distribution



Question 1

All the digits 1 to 9 are to be placed in the circles in Figure 1, one in each, so that the total of the numbers in any line of four circles is the same. In the example shown in Figure 2, the total is equal to 20.

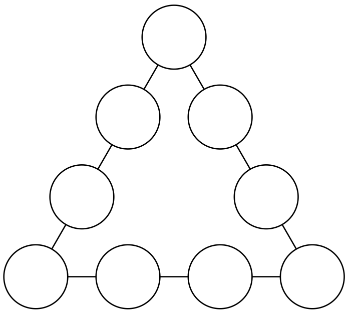


Figure 1

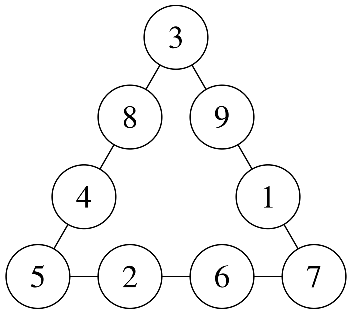


Figure 2

Prove that if the total T is possible then the total $40 - T$ is possible.

Solution

Consider a numbering in which the total is T. In each circle replace the number d with the number $10 - d$. The digits 1 to 9 will still all be used once each, and now each line has total $40 - T$.

Alternative

This alternative solution involves finding (with proof!) all possible totals and then noting that they come in pairs $(T, 40 - T)$.

The sum of all nine numbers is $1 + 2 + \cdots + 9 = 45$. This should equal $3T$ minus the sum of three corner numbers which are counted twice.

Denote the corner numbers a, b and c. Then

$$3T - (a + b + c) = 45 \implies 3T = 45 + (a + b + c).$$

Since a, b and c are all different numbers between 1 and 9, the smallest and largest possible values of $a + b + c$ are 6 and 24, respectively. This means that $17 \leq T \leq 23$.

The question gives an example with $T = 20$. It is possible to construct examples with $T = 17, 19, 21$ and 23. (A full solution should show these examples.)

To get $T = 18$ we need $a + b + c = 9 = 1 + 2 + 6 = 1 + 3 + 5 = 2 + 3 + 4$. For each of those options there are only a few possibilities to check, and none of them give a valid configuration. So $T = 18$ is not possible. A similar analysis shows that $T = 22$ is not possible either. (Again, a full solution needs to show that all the possibilities have been tested.)

In conclusion, the only possible totals are $17, 19, 20, 21$ and 23, and hence the claim of the question if true.

Markers' comments

Many candidates attempted this question, although not as many as might be expected from a first problem.

The most common approach was to derive an expression such as $3T = 45 + (a + b + c)$ and from there put bounds on T. However many candidates, having reached this point, fell short of a full solution by stating that all values of T in this range had a corresponding $40 - T$, and did not go on to show which of these totals were possible.

Some candidates proved that $T = 18$ was impossible, and from this claimed that $T = 22$ must also be, otherwise the question itself would be wrong. But this uses precisely the fact that you are required to prove - indeed, in this case it is *crucial* to check that $T = 22$ is also impossible.

A good approach sometimes seen was to rearrange $3(40 - T) = 45 + (a + b + c)$ into a form that suggested the corner numbers should be replaced by $10 - a$, $10 - b$, $10 - c$. From here candidates who spotted a substitution such as every digit d going to $10 - d$ that achieved this were awarded full credit, whereas candidates who were unable to explain where the digits $10 - a$, $10 - b$, $10 - c$ could be sourced were not.

Of those who solved the problem using the $T - d$ substitution many lost a mark for not checking that the resulting configuration satisfies the conditions of the question (namely, that the numbers are all different and between 1 and 9).

Question 2

A positive integer is said to be *jiggly* if it has four digits, all non-zero, and no matter how you arrange those digits you always obtain a multiple of 12.

How many jiggly positive integers are there?

Solution

Since rearrangement always gives an even integer, all the digits must be even. Numbers ending '$a2$' or '$a6$' are not divisible by 4 when a is even. Therefore all the digits must be multiples of 4, that is, every digit is either 4 or 8. A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Therefore, if the number is to be divisible by 12 then neither 4 nor 8 can occur 3 times or 4 times, and so it is an arrangement of 4488, of which there are six (4488, 4848, 4884, 8448, 8484 and 8844). Thus there are precisely six jiggly positive integers.

Note

In this case, the number of arrangements of the digits 4488 is sufficiently small for you to write out and count all of them. If you know about combinatorics, you can also calculate it as $\frac{4!}{2!2!}$.

Markers' comments

This question attracted a response from almost every candidate and many achieved full marks, however it was common for a response to be awarded 4 marks even though the correct answer was obtained. This is in accordance with the instructions at the front of the question paper, that a majority of marks are given to explanations and justifications, rather just the answer.

Often candidates' first claim was that each of the four digits must be even and as obvious as this may seem some reasoning was required. In many responses the next claim made was that the digits 2 and 6 could not be used. This is a vital step towards the final solution and as such a clear argument was required to back up the claim. A candidate could have stated that all the digits are even and when the number under consideration is halved all of its digits may be halved. As a jiggly number must be a multiple of four it should still be even after halving, no matter the order of the digits, and so the digits 2 and 6 must not have been present to begin with. An alternative argument would be to consider the criterion for divisibility by four and list all two-digit multiples of four, identifying that only 44, 48, 84 and 88 can be reversed.

Quite a few candidates seemed to think that if a number is multiple of 12 then the two-digit numbers that make it up must be multiples of 12. This is not the case; for example, 4644 is a multiple of 12 although 46 and 44 are not.

Some candidates considered only divisibility by three and although this reduced the number of four-digit integers under consideration substantially, care was needed to list those that remained. Where a method of exhaustion was used it was successful when options where listed systematically. Some candidates listed options haphazardly and as a result missed certain cases, something that was penalised heavily as the method of exhaustion requires every possibility to be considered.

Question 3

Four different points A, B, C and D lie on the curve with equation $y = x^2$.
Prove that $ABCD$ is *never* a parallelogram.

Solution

Let the points A, B, C, D be (a, a^2), (b, b^2), (c, c^2), (d, d^2) respectively, where, without loss of generality, $a < b < c < d$. We investigate which of the sides through A could be parallel to another side. The slope of the line-segment AB is $(b^2 - a^2)/(b - a)$, which is $a + b$. The slopes of AC, AD, BC, BD, and CD are given by the analogous expressions. Now $a + b < c + d$ and $a + c < b + d$ and so $AB \nparallel CD$ and $AC \nparallel BD$.

Thus at most one side through A (namely AD) can be parallel to another side, and therefore $ABCD$ is not a parallelogram.

Alternative

Let the coordinates of the four points be defined as above. If $ABCD$ is a parallelogram then AB is parallel to DC and BC is parallel to DA.

From the calculation of gradients given above, this means that $a + b = c + d$ and $b + c = a + d$. Subtracting the second equation from the first gives $a - c = c - a$ which implies that $a = c$. But this is not possible, as the four points are different.

Thus our initial assumption that two pairs of sides are parallel must be false, so $ABCD$ is not a parallelogram.

Alternative

Although the surest way to obtain full marks on this question is by calculation, it is possible to produce an argument that only uses the sign of the gradient. Such approaches often failed to be fully rigorous because candidates did not consider all possible cases. However, the markers were pleased to see that several candidates did manage to produce a full solution using this approach. Here we give a simplified version of their arguments.

Let the points A, B, C, D have coordinates $(a, a^2), (b, b^2), (c, c^2), (d, d^2)$ with $a < b < c < d$. Suppose first that a^2 is the smallest. Then we must have $a^2 \leq b^2 < c^2 < d^2$. This means that AB, BC and CD all have positive gradients, and so $\angle ABC$ and $\angle BCD$ are both obtuse (being the angle between two positive-gradient lines). But it's impossible for adjacent angles in a parallelogram to be obtuse.

Suppose instead that b^2 is the smallest (and a^2 is not); we then have $a^2 > b^2 \leq c^2 < d^2$. Then AB has negative gradient and CD has positive gradient, so they cannot be parallel.

The cases where c^2 or d^2 are the smallest are just reflected versions of these. The inequalities have been chosen above to cover the cases where two of the y-coordinates tie for the minimum, so this covers all cases.

Markers' comments

A good number of candidates attempted this problem. Some went down the calculation route, finding gradients or lengths. These were sometimes successful, but often considered only special cases (such as $a = -b$) or made the algebra too complicated to complete the proof.

A lot of candidates had convinced themselves but were not able to construct a sufficiently mathematically rigorous argument to get high marks. After drawing a few diagrams it seems very hard to make a parallelogram, but you have to be sure that the diagrams you draw cover all cases. For example, if two points are on either side of the y-axis (say when A, B, C, D are in that order left-to-right), then AB has negative gradient whilst CD has positive gradient, so clearly a parallelogram cannot exist in this case. But most candidates attempting solutions like this didn't find something rigorous in the case where all four points are on the same side, say. Those who were successful mostly used the angles argument outlined above.

Question 4

Let n be an odd integer greater than 3 and let $M = n^2 + 2n - 7$.

Prove that, for all such n, at least four different positive integers (excluding 1 and M) divide M exactly.

Solution

Since n is odd and $n > 3$ we may write $n = 2k + 1$ where k is an integer and $k \geq 2$. Then $M = 4k^2 + 8k - 4 = 4(k^2 + 2k - 1)$. Even if $k^2 + 2k - 1$ is prime, we see that M has the four positive integral divisors 2, 4, $k^2 + 2k - 1$ and $2(k^2 + 2k - 1)$, all of them different from 1 and M. Clearly they are all different since (given that $k \geq 2$) $k^2 + 2k - 1 \geq 7$.

Alternative

A number of candidates produced the following elegant argument. Completing the square gives $M = (n + 1)^2 - 8$. Since n is odd, $n + 1$ is even so $(n + 1)^2$ is a multiple of 4, and subtracting 8 always leaves a multiple of 4. Hence M is divisible by 2, $\frac{M}{2}$, 4 and $\frac{M}{4}$. Since $n \geq 5$, $M \geq 28$ so $\frac{M}{4} \geq 7$ and hence the four factors are all different (and different from 1 and M).

Alternative

The conclusion that M is a multiple of 4 can also be reached using modular arithmetic. Since n is odd, $n \equiv \pm 1 \pmod 4$ so $n^2 \equiv 1 \pmod 4$. Hence $M \equiv 1 \pm 2 - 7 \equiv -4$ or $-8 \equiv 0 \pmod 4$.

Note

When $k^2 + 2k - 1$ is prime (as happens, for example, when k is 2 or 4 or 6 or 8), M has *exactly* four different factors (excluding 1 and itself). But it can have more than four different factors: for example, when k is 3, M is divisible by $2, 4, 7, 8, 14$ and 28. These examples show that every algebraic factor of an expression gives a numerical factor, but there may be numerical factors that don't come from an algebraic factor.

Markers' comments

This was the best done question in terms of the candidates scoring near full marks.

Many students tackled the question by setting $n = 2k + 1$ (where $k \geq 2$) or $n = 2k - 1$ (where $k \geq 3$) or $n = 2k + 3$ (where $k \geq 0$). Others used remainders on division by 4 to show that M is of the form $4K$ for some positive integer K. There were some impressive solutions. Some lost a mark or two through not writing down four divisors explicitly, or not showing explicitly that the divisors (such as 4 and K) that they had found were different from each other and from 1 and M.

Some students proved that 2 and 4 are always divisors of M but did not seem to realise that so are $\frac{M}{2}$ and $\frac{M}{4}$. This could be because they thought they were looking for other numbers which were divisors of every such M, rather than allowing different divisors for different M's.

Some students thought that n must be of form $4k + 1$, or even $2k^2 + 1$. Although such numbers are odd and greater than 3, not every number that is odd and greater than 3 is of such a form, so such an approach scored few marks.

Question 5

Claire and Stuart play a game called *Nifty Nines*:

(i) they take turns to choose one number at a time, with Claire choosing first;
(ii) numbers can only be chosen from the integers 1 to 5 inclusive;
(iii) the game ends when n numbers have been chosen (repetitions are permitted).

Stuart wins the game if the sum of the chosen numbers is a multiple of 9, otherwise Claire wins.

Find all values of n for which Claire can ensure a win, whatever Stuart's choices were. You must prove that you have found them all.

Solution

Let s_k be the sum of the numbers chosen up to and including the k^{th} step, and let r_k be the remainder when s_k is divided by 9. If Claire makes the last choice, she can choose the number 1 or 2 to ensure that $r_n \neq 0$, so that s_n is not a multiple of 9. She makes the last choice if and only if n is odd; therefore if n is odd then Claire can force a win.

She can also ensure that she wins if n is even and not a multiple of 3, that is, n is of one of the forms $6m + 2$, $6m + 4$ for some m. To do this, on her first move Claire chooses 3. Thereafter, at step $2k + 1$, if Stuart has chosen d at stage $2k$ then Claire chooses the number $6 - d$. If $n = 6m + 2$, after $3m$ pairs of steps s_{6m+1} will be $3 + 3m \times 6$, so $r_{6m+1} = 3$ and Stuart will be unable to make a multiple of 9 at the n^{th} step; and if $n = 6m + 4$, after $3m + 1$ pairs of steps s_{6m+3} will be $3 + (3m + 1) \times 6$, so $r_{6m+3} = 0$ and again Stuart will be unable to make a multiple of 9 at the n^{th} step.

If, however, n is a multiple of 6 then Stuart can ensure that he wins: whatever number d Claire chooses at stage r (where $r < n$ and r is odd), Stuart chooses $6 - d$ at stage $r + 1$. This ensures that $s_{k+6} = s_k + 18$ and Stuart can thereby ensure that $r_k = 0$ whenever k is a multiple of 6.

The conclusion is that Claire can force a win if n is not a multiple of 6; she cannot force a win if n is a multiple of 6.

Markers' comments

Few candidates achieved more on this problem than an understanding of the case of odd n or specific small values of n.

When working with a game, it is important to understand that each player may make their moves in response to the other player's moves; thus, you are looking for a strategy for one player that says how to respond to any choice the other player makes.

It is possible to determine the winner by working backwards from the last move. Supposing Stuart has the last move, you can identify which values of the sum s_{n-1} before that last move allow him to win, and which must result in a win for Claire. You can then consider what values of s_{n-2} allow Claire to leave a value of s_{n-1} with which Stuart cannot win, and so on. Such an approach ends up proving more than is needed to solve the problem, and is more complicated to write up, but having determined the winner in that way it is also possible to extract a winning strategy from the backtracking argument.