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Mathematical Olympiad for Girls 2018

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

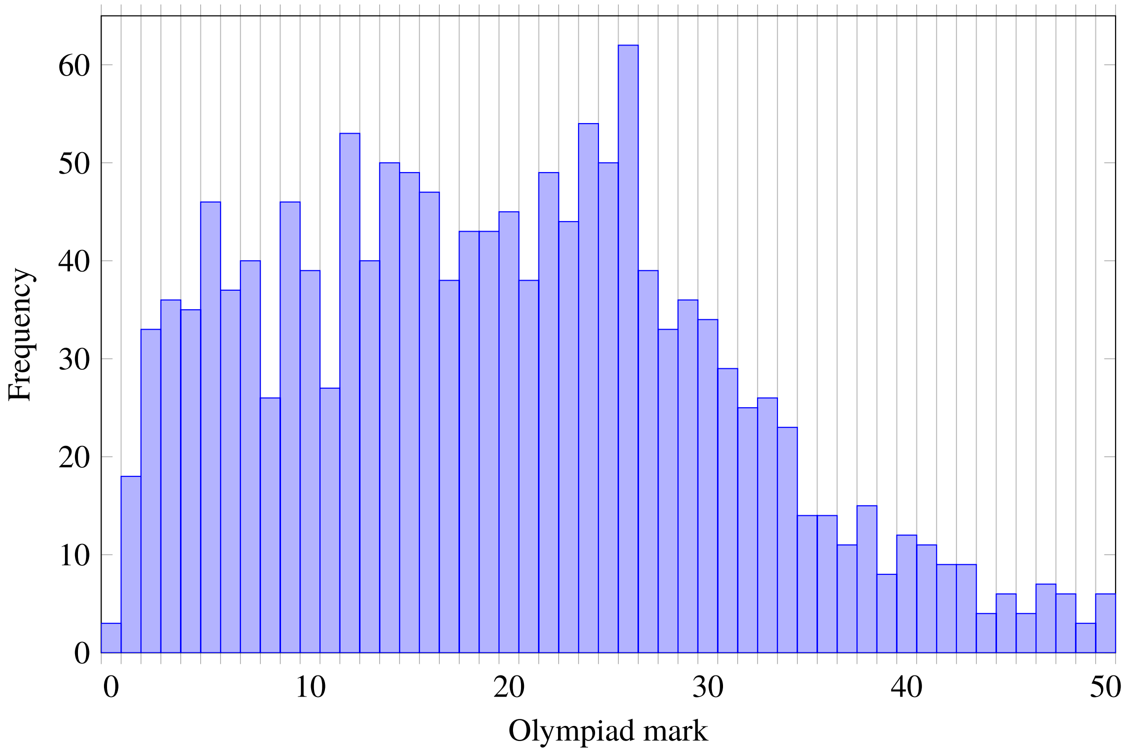
The markers were extremely pleased to see a high level of engagement with the paper, with very few answers left blank, even on the later questions. Questions 4 and 5 produced some good attempts, even if reaching complete solutions proved too challenging for most. Candidates should recall the marking principles described above and understand that scoring 2 or 3 marks on the last two questions is a great achievement.

Questions 1 and 2 produced some excellent, well explained answers, with over half the candidates solving the problems fully. Marks were often lost for lack of explanation – for example, not showing how the factors found related to the factorisation in Question 1, or confusing terms 'radius' and 'tangent' in Question 2.

The introductory parts in Questions 1, 3 and 5 were intended to help with the strategy for the main problem. Those were well utilised in Questions 1 and 5, but less so in Question 3. Candidates in future years should look out for solutions which use the result from the introductory part.

The 2018 Mathematical Olympiad for Girls attracted 1475 entries. The scripts were marked on 13th and 14th October in Cambridge by a team of Alexander Gunning, Andrew Carlotti, Carol Gainlall, Chris Eagle, Constance Bambridge-Sutton, Dominic Yeo, Abhilasha Aggarwal, Emiko Okoturo, Emily Beatty, Eve Pound, Georgina Majury, James Cranch, James Gazet, James Harris, Jerome Watson, John Haslegrave, Joseph Myers, Kamran Pentland, Kasia Warburton, Lynn Walton, Martin Orr, Philip Coggins, Sayan Biswas, Stephen Tate, Sue Cubbon, Tim Cross, Tom Bowler and Vesna Kadelburg.

Mark distribution



Question 1

(a) Write down the full factorisation of the expression $a^2 - b^2$.

(i) Show that 359999 is not prime.

(ii) Show that 249919 is not prime.

Hint You can use your factorisation of $a^2 - b^2$ if you like. (5 marks)

(b) Write down the full factorisation of $a^2 + 2ab + b^2$.

(i) Show that 9006001 is not prime.

(ii) Show that 11449 is not prime.

(5 marks)

Solution

(a) $a^2 - b^2 = (a - b)(a + b)$.

(i) $359999 = 360000 - 1 = 600^2 - 1^2 = 599 \times 601$, so 599 and 601 are factors of 359999, and so 359999 is not prime.

(ii) $249919 = 250000 - 81 = 500^2 - 9^2 = 491 \times 509$, so 491 and 509 are factors of 249919, and so 249919 is not prime.

(b) $a^2 + 2ab + b^2 = (a + b)^2$.

(i) $9006001 = 9000000 + 6000 + 1 = 3000^2 + 2 \times 3000 \times 1 + 1^2 = 3001^2$, so 3001 is a factor of 9006001, and so 9006001 is not prime.

(ii) $11449 = 10000 + 1400 + 49 = 100^2 + 2 \times 100 \times 7 + 7^2 = 107^2$, so 107 is a factor of 11449, and so 11449 is not prime.

Markers' comments

This question was found accessible by a majority of candidates and it was excellent to read many clearly explained answers. As the main part of this question was numerical in nature, it was expected that some justification of the factors chosen was given: this included either a direct calculation of a multiplication or an indication of how the large numbers were split up into either sums or differences to indicate how the algebraic factorisation was being utilised. Some indication needed to be provided in the answer to link the factorisations back to the question about primality.

Question 2

Triangle ABC is isosceles, with $AB = BC = 1$ and angle ABC equal to $120°$. A circle is tangent to the line AB at A and to the line BC at C.

What is the radius of the circle? (10 marks)

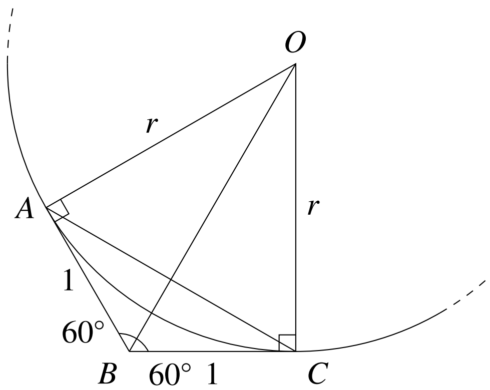
[*You should state clearly any geometrical facts or theorems you use in each step of your calculation. For example, if one of your steps calculates the size of an angle in a triangle you might justify that particular step with "because angles in a triangle add up to* 180 *degrees."*]

Solution

Label the centre of the circle O and the radius of the circle r. Then angle $OAB = 90°$, since a tangent to a circle is perpendicular to the radius at the point of contact.

Also, by symmetry of the diagram, the line BO bisects the angle ABC.

Now consider triangle OAB. It has a right angle at A, angle $ABO = 60°$, $AB = 1$ and $OA = r$. Therefore $\frac{r}{1} = \tan(60°) = \sqrt{3}$ and so $r = \sqrt{3}$.



Note

In the above solution, we stated that the line BO bisects the angle ABC, 'by symmetry'. This is an acceptable assertion in this case but, strictly speaking, you should prove it. This can be done by noting that triangles OAB and OCB have three equal sides and are therefore congruent.

Alternative

We can use the cosine rule in triangle ABC, knowing $\cos(120°) = -\frac{1}{2}$, to find $AC = \sqrt{3}$. Then note that angle $AOC = 360 - 90 - 90 = 60°$ (using the sum of the angles in the quadrilateral $OABC$), so triangle AOC is equilateral, and hence $r = \sqrt{3}$.

Markers' comments

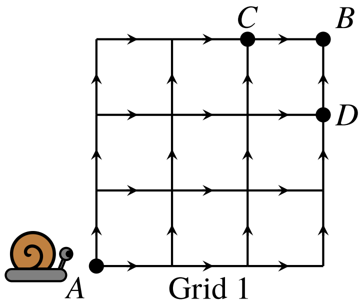
It was good to see that so many candidates made sensible progress on this question, being able to find AC by trigonometry or similar methods, and recognising the relationship between radius and tangent. However, some struggled with drawing the correct diagram, confusing "tangent" with "radius", or assuming AC was the diameter, and so ended up calculating the wrong length.

Over half the candidates described a clear idea of how to find the radius from AC. Of those, a few people slipped up in the statement of the sine or cosine rule; more commonly marks were lost for unsimplified answers in terms of trigonometric ratios, or misquotings of these values. Students who avoided trigonometry entirely and identified triangles as halves of equilateral triangles generally avoided this problem.

Question 3

(a) Sheila the snail leaves a trail behind her as she moves along gridlines in Grid 1. She may only move in one direction along a gridline, indicated by arrows. Let b, c, d be the number of different trails Sheila could make while moving from A to B, C, D respectively.

Explain why $b = c + d$. (2 marks)



Grid 1

(b) Ghastly the ghost lives in a haunted mansion with 27 rooms arranged in a $3 \times 3 \times 3$ cube. He may pass unhindered between adjacent rooms, moving through the walls or ceilings. He wants to move from the room in the bottom left corner of the building to the room farthest away in the top right corner, passing through as few rooms as possible. Unfortunately, a trap has been placed in the room at the centre of the house and he must avoid it at all costs.

How many distinct paths through the house can he take? (8 marks)

Solution

(a) To get to B, Sheila can either go to C and then move to the right, or go to D and then move up. Hence the number of trails she can make while moving from A to B equals the number of trails from A to C plus the number of trails from A to D: $b = c + d$, as required.

(b) We can represent the rooms in the mansion by cells in three 3×3 grids. Suppose that Ghastly wants to move from the room marked S to the room marked F. In order to take the shortest possible path, he should only move to the right, towards the back, or up.

For each room, the number of ways he can get to that room equals the sum of the number of ways he can get to the rooms he could have visited immediately before it; those are the rooms to the left, in front, or below the current room (in some cases, not all three of those rooms exist).

We can therefore find the number of ways to get to each rooms by filling in the cells in the tables, starting from S. There is only one way to get to the three rooms adjacent to S – that is, to come straight from S. From there, we can fill in the tables as shown below. Note that Ghastly cannot go to the central room, so we place a 0 there.

1	3	6
1	2	3
S	1	1

3	6	18
2	0	6
1	2	3

6	18	F
3	6	18
1	3	6

From the table, the total number of ways to get to room F is $18 + 18 + 18 = 54$.

Alternative

Another approach for part (b) is to count the number of paths between the two rooms without regard for the trap, then to subtract the number of paths that go through the trap.

A path must have two moves right, two upward and two back. If all six of those moves were distinguishable, there would be $6! = 720$ possible orderings of those moves. However, the two moves right are indistinguishable, and likewise the two moves upward and the two moves back. So there are $6!/2!^3 = 90$ paths, including those that go through the trap. (This is a *trinomial coefficient*.)

A path from the starting room to the trap has one move right, one upward and one back, so there are $3! = 6$ such paths, and similarly there are $3! = 6$ moves from the trap to the final room. So there are $3!^2 = 36$ paths that go through the trap, and so $90 - 36 = 54$ paths that do not go through the trap.

Markers' comments

This question proved the most challenging in terms of making progress towards a viable strategy to count the number of required paths. Although over half the candidates were able to explain the result in part (a) clearly, not many saw how to use the idea in part (b). Even those who realised that F is the sum of the numbers at the three adjacent vertices often decided to find those three numbers by listing all possible paths rather than continuing with the same strategy working backwards to the start; such attempts were sometimes successful, although counting mistakes were common.

The most common approach was to categorise the paths by looking at which rooms Ghastly uses to go up a level, and then either count all the paths which avoid the trap, or count all possible paths and then subtract those that go through the trap. This approach needs a careful strategy to ensure no paths are missed. Successful candidates did a very good job of laying out their working and explanations clearly.

Question 4

Each of 100 houses in a row are to be painted white or yellow. The residents are quite particular and request that no three neighbouring houses are all the same colour.

(a) Explain why no more than 67 houses can be painted yellow. (4 marks)

(b) In how many different ways may the houses be painted if exactly 67 are painted yellow? (6 marks)

Solution

We shall denote a house that is to be painted white by W and a house that is to be painted yellow by Y.

(a) Let us number the houses 1 to 100 from left to right and consider the 34 blocks $(1), (2, 3, 4), (5, 6, 7), \ldots, (95, 96, 97), (98, 99, 100)$. As no three neighbouring houses can all be the same colour there must be a maximum of two yellow houses in each of the 33 blocks of three houses. From this we can deduce that at most $1 + 2 \times 33 = 67$ houses could be painted yellow.

Note

It is possible to paint exactly 67 houses yellow, one colouring that achieves this is Y followed by 33 blocks of WYY.

(b) Each block of three houses could be painted YYW, YWY or WYY. Note that the second colouring cannot be followed by the first and the third colouring cannot be followed by either the first or the second. This means that as soon as we choose the third colouring for one of our blocks of three houses then all successive blocks must have the same colouring. The first house must be painted Y, as demonstrated in part (a), and the next block of three could be painted YWY or WYY. The only choice we have is when we first paint a block WYY, this could be in any of the 33 blocks of three houses or not at all. This means there are 34 different ways to paint the 100 houses, which adhere to the strict requests of the residents.

Markers' comments

The vast majority of candidates made some progress on this question, particularly in part (a), yet it yielded the least number of full mark solutions. For part (a) it was common for candidates to assume that the first 99 houses must be painted with blocks of YYW and then to argue that as 66 houses were painted yellow and the final house may also be painted yellow then 67 was the maximum. Unfortunately, as referred to in part (b), there were other colourings possible so such an explanation was not general enough. It is important not to specify a particular colouring when trying to establish that no more than 67 houses could be painted yellow. Many candidates gave an example to show that exactly 67 houses could be painted yellow, but in fact this was unnecessary given the phrasing of the question.

A common incorrect answer to part (b) was two colourings, this often arose due to candidates believing that the first block of three houses could be painted YYW or YWY and that subsequent blocks must be painted in exactly the same way as the first. This is not correct as, for example, YYW can be followed by YWY so the different block types can interact with one another, this simple but vital observation was often overlooked. Some candidates attempted to argue that because there were 33 pairs of consecutive yellow houses and a single yellow house there must be 34 different colourings, each arising from a different position for the single yellow house. This argument can be made rigorous but in order to be awarded full marks a candidate must explain clearly why all possible colourings fit the criteria of having 33 pairs of consecutive yellow houses and a single yellow house. The easiest way to do this is to consider how the 67 yellow houses can be placed in to the gaps between the 33 white houses (given the two ends there are in fact 34 such gaps).

Question 5

Sophie lays out 9 coins in a 3×3 square grid, one in each cell, so that each coin is tail side up. A move consists of choosing a coin and turning over all coins which are adjacent to the chosen coin. For example if the centre coin is chosen then the four coins in cells above, below, left and right of it would be turned over.

(a) Sophie records the number of times she has chosen each coin in a 3×3 table. Explain how she can use this table to determine which way up every coin in the grid is at the end of a sequence of moves. (2 marks)

(b) Is it possible that after a sequence of moves all coins are tail side down? (4 marks)

(c) If instead Sophie lays out 16 coins in the cells of a 4×4 grid, so that each coin is tail side up, is it possible that after a sequence of moves all coins are tail side down? (4 marks)

[*In parts* (b) *and* (c)*, if you think that it is possible, you should specify a sequence of moves, after which all coins are tail side down. If you think it is not possible, you should give a proof to show that it can't be done, no matter which sequence of moves Sophie chooses to do.*]

Solution

(a) The number of times that a particular coin C has been turned over is equal to the total number of times that a coin in a cell adjacent to C has chosen. So, to find the end position of C, Sophie can add together all of the numbers in the table representing the cells adjacent to C to get C_{total} and turn C over C_{total} times. Doing this for each coin gives the end position of all of the coins in the grid.

(b) It is not possible. Suppose that it can be done. Let a, b, c, d be the entries in Sophie's table as shown in the table below. For a particular coin C, if C_{total} is even, C will be tail side up, and if it is odd, C will be tail side down, since turning a coin over twice returns it to its original state. So, since the centre coin ends tail side down, we must have $a + b + c + d$ odd. Since the coins in the top left and bottom right end up tail side down, we must have $a + b$ and $c + d$ odd. But then $(a + b) + (c + d)$ is a sum of two odd numbers, which is even. But we already know that $a + b + c + d$ is odd, and it can't be both odd and even, so the initial assumption that all of the coins can be tail side down after a series of moves must be wrong.

	a	
b		c
	d	

(c) It is possible, for example in the following table, every coin is in a cell adjacent to exactly one other cell in which a coin has been chosen, so every coin is turned over exactly once; Sophie could choose each of the six coins in cells with a 1 once in any order.

0	1	1	0
0	0	0	0
1	0	0	1
1	0	0	1

Alternative

In the following alternative solution to part (b) the term *parity* will be used, the *parity* of a number or variable refers to whether it is odd or even.

Let T be the number of coins in the main diagonal, running from top left to bottom right, which are tail side down. Initially $T = 0$ and if all the coins are to be tail side down then T would take the value 3, which we will show is impossible.

Considering the square grid we can see that any move that changes the state of a single coin in the main diagonal will in fact change the state of exactly two coins in the main diagonal, it is impossible to change the state of 1 or 3 of these coins in a single move. If we consider two of the coins in the main diagonal which may be affected by a single move then there are three distinct cases, both coins may be tail side up, both may be tail side down or one tail side up and the other tail side down. In each of the three cases a single move would preserve the parity of the number of coins tail side down and hence preserve the parity of T. We deduce that $T = 3$ is not possible and hence it is impossible for all 9 coins to be tail side down.

Note

The above solution generalises to any square grid of size $(2n + 1) \times (2n + 1)$, where n is a positive integer.

Markers' comments

This was the least popular question, but those who attempted it often made good progress in at least one of the parts. Many picked up on the hint from part (a) and looked at the sum of the times that the coins adjacent to a particular coin were chosen.

In part (a) it was necessary both to describe how the state of any particular coin can be read from the table, and to explain why this works.Some candidates tried to argue that Sophie can find out the state of the board by "simulating" the game, choosing each coin the numbers of time shown in the table. However, for this to work it is necessary that the order of the moves does not matter. This turns out to be the case, but requires some work to prove.

Part (b) required an explanation of why the required end-position is impossible to achieve regardless of the number of times each coin is chosen. A solution that refers to a specific example is therefore unlikely to score full marks. Successful solutions usually looked at the middle and the four corner coins.

A common approach was to look at how selecting a particular coin affects neighbouring coins (rather than how each coin is affected by its neighbours). This approach is much less likely to lead to a full explanation.

Part (c) required an example with a justification why it works. It was pleasing to see so many candidates attempting to find an example even if they did not manage to solve part (b).

A relatively large number of candidates misinterpreted the question, thinking that the chosen coin is turned over along with all its neighbours. With this interpretation, the answer to both parts (b) and (c) is 'Yes' which makes the question a little easier. However, some of the ideas needed for the correct solution are still useful, so up to five marks were available in this case.