United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

The mark allocation on Mathematics Olympiad papers is different from what you are used to at school. To get any marks, you need to make significant progress towards the solution. This is why the rubric encourages candidates to try to finish whole questions rather than attempting lots of disconnected parts.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or been unclear in your explanation, then you will normally receive nearly all the marks.

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1. At Mathsland Animal Shelter there are only cats and dogs. Unfortunately, one day 60 of the animals managed to escape. Once a volunteer had realised, they counted the remaining animals. They noted that half of the cats and a third of the dogs had escaped.

(a) (i) If the number of cats before the escape was C and the number of dogs before the escape was D, write down an equation linking C and D.

(ii) If the total number of animals before the escape was T, write down an equation linking C, D and T.

(4 marks)

(b) Given that more cats than dogs escaped, find the largest possible value of T. You must justify why the value you have found is the largest. (6 marks)

Solution

Commentary

This is similar to standard simultaneous equations questions you have probably met at school. The difference is that, once you have written down the two equations, you will see that you don't have enough information to uniquely determine the values of C, D and T.

You may want to start by trying to find the values of C and D for various values of T. Can you always find a solution that works? Remember that the question places some constraints on possible values of C and D.

To justify that the value you have found for T is the largest possible, you must show that this value can be achieved (by showing an example of C and D in that case) and also that any value larger that is cannot be achieved.

(a) The total number of escaped animals is 60, and the total number of animals before the escape is T.

(i) $\frac{1}{2}C + \frac{1}{3}D = 60$

(ii) $C + D = T$

(b) From the first equation, $D = 180 - \frac{3}{2}C$. Substituting into the second equation,

$$T = C + (180 - \tfrac{3}{2}C) = 180 - \tfrac{1}{2}C.$$

Therefore T is the largest possible when C is the smallest possible. Since more half of the 60 escaped animals were cats, $\frac{1}{2}C \geq 31$, so $T \leq 180 - 31 = 149$. This value is achieved with $C = 62$ and $D = 87$.

Hence the largest possible value of T is 149.

Commentary

An alternative approach for part (b) is to solve the equations for C and D in terms of T.

Rewrite the first equation as $3C + 2D = 360$. Multiplying the second equation by 2 and subtracting from the first gives $C = 360 - 2T$. Substituting back then gives $D = 3T - 360$.

You now need to consider the constraints on C and D. First, both need to be non-negative, so $T \leq 180$ and $T \geq 120$. The condition that more cats and dogs escaped means that $\frac{1}{2}C > \frac{1}{3}D$ or, equivalently, $3C > 2D$. Substituting the expressions for C and D in terms of T gives:

$$3(360 - 2T) > 2(3T - 360).$$

Rearranging this inequality gives $T < 150$ so, since T is an integer, the largest possible value of T is 149. This value is achieved when $C = 62$ and $D = 87$, which satisfy the conditions of the problem.

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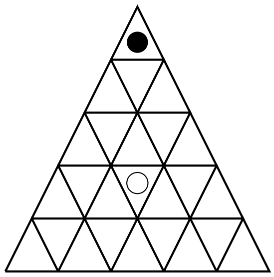
2. Beth has a black counter and Wendy has a white counter. Beth and Wendy move their counters on the two boards below according to the starting positions and rules given. They always move their counters at the same time.

(a) 

At each turn, each player moves their counter either one square to the left or one square to the right. Prove that the black and white counters can never be in the same square at the same time.

Hint You may find it helpful to refer to the colours of the squares on the board in your explanation. (3 marks)

(b) At each turn, each player moves their counter to a triangular cell which shares one edge with the cell that their counter is currently in. Can their counters ever be in the same cell at the same time? (7 marks)



Hint If you think the two counters can never be in the same cell at the same time, you should give an argument that they cannot be in the same cell at the same time which works no matter which sequence of moves Beth and Wendy do. If you think the two counters can be in the same cell at the same time, you should give an example of a sequence of moves after which they are in the same cell at the same time.

Solution

Commentary

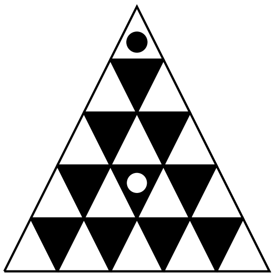
In a problem like this, it is tempting to try to think of all possible ways that the two counters can move and find a “worst possible” or “best possible” sequence of moves. Another strategy would be to consider where the counters were the move before they were in the same square and then “work backwards” to the starting position. It turns out that, in many cases, both of these strategies are either impractical (because there are too many possible sequence of moves) or impossible (because, like here, the counters can keep moving forever without meeting).

Instead, you need to think of some property of the game that means that the counters can never be in the same square. The hint in part (a) suggests thinking about how the colour of each counter’s square changes with each move.

Once you have seen that considering the colours of the cells is useful in part (a), you may want to try a similar strategy in part (b). You need to decide how to colour the triangular cells so that each move changes the colour of the counter’s cell. You also need to show that, with your colouring, the black and the white counter start on different colours.

(a) Since the colours of the squares on the board alternate, each move (one square to the left or one square to the right) changes the colour of the square that the counter is in. The two counters start on different colours so, since each one changes the colour at each turn, they will always be on different colours. This means that they can never be in the same square at the same time.

(b) Colour the triangular cells black and white as shown in the diagram.



Then two cells which share an edge are different colours. Hence each move changes the colour of the counter's cell.

As can be seen in the diagram, the black and the white counter start on different colours. Therefore they will always be on different colours, and so cannot be in the same cell at the same time.

3. (a) Seth wants to know how many positive whole numbers from one to one hundred are divisible by two or five. He thinks that the answer is 70 because there are fifty multiples of two and twenty multiples of five from one to one hundred. Explain why his answer is too large. (2 marks)

(b) Consider the list of 1800 fractions

$$\frac{1}{1800}, \frac{2}{1800}, \ldots, \frac{1799}{1800}, \frac{1800}{1800}.$$

How many are *not* in simplest form? Explain your reasoning. (8 marks)

[Note: The fraction $\frac{900}{1800}$ is not in simplest form because it can be simplified to $\frac{1}{2}$.]

Solution

Commentary

The main part of the question (part (b)) asks how many of the fractions can be simplified. How can you tell when a fraction can be simplified? Part (a) should help you avoid a common trap when counting multiples.

It may help to start by writing out some multiples of 2 and 5. Which numbers will be listed more than once? This tells you not only why 70 is too large, but also helps you find the correct answer to Seth's question.

In part (b) you need to think carefully what exactly you want to count. For example, 1800 is divisible by both 3 and 9, but do you need to count the multiples of 3 and 9 separately?

You also need to extend the reasoning from part (a) to avoid Seth's trap. You may find it helpful to use a Venn diagram to represent the number of multiples.

(a) If we list all 50 multiples of 2 and all 20 multiples of 5, all the multiples of 10 will appear in both lists. Since some numbers are counted twice, the real answer is smaller than 70.

(b) A fraction will not be in simplest form when the numerator shares at least one factor with 1800. Since $1800 = 2^3 \times 3^2 \times 5^2$, we need to count how many of the numbers from 1 to 1800 are divisible by 2, 3 or 5.

Imagine making three separate lists: one containing the multiples of 2, one containing the multiples of 3 and one containing the multiples of 5. There are $\frac{1800}{2} = 900$ numbers in the first list, $\frac{1800}{3} = 600$ in the second and $\frac{1800}{5} = 360$ in the third, a total of 1860 numbers.

The multiples of 6, 10 and 15 appear twice, so we need to take away $\frac{1800}{6} + \frac{1800}{10} + \frac{1800}{15} = 600$.

However, the multiples of 30 have now been taken away three times. But they appear three times in the original three lists, so they should have only been taken away twice. Hence we need to add back $\frac{1800}{30} = 60$.

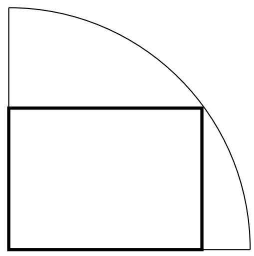
The required total is therefore $1860 - 600 + 60 = 1320$. Thus 1320 of the fractions are not in simplest form.

Commentary

Try drawing a Venn diagram to show the number of multiples of 2, 3 and 5. Does that make our calculation clearer?

This idea can be extended to count the total number of elements in more than three overlapping sets, resulting in what is know as the inclusion-exclusion principle.

4. The diagram shows a rectangle placed inside a quarter circle of radius 1, such that its vertices all lie on the perimeter of the quarter circle and one vertex coincides with the centre of the (whole) circle.



Let the perimeter of such a rectangle be P.

(a) Show that $P = 3$ is impossible. (4 marks)

(b) Find the largest possible value of P. You must fully justify why the value that you find is the largest. (4 marks)

Instead a rectangle is placed inside a whole circle of radius 1, such that its vertices all lie on the circumference of the circle.

(c) If the perimeter of the rectangle is as large as possible, show that the rectangle *must* be a square and calculate its perimeter. (2 marks)

Solution

Commentary

Call the sides of the rectangle x and y. It seems reasonable to start by writing some equations connecting x, y and P. Looking at parts (a) and (b) together, it sounds like these equations will only have a solution for some values of P.

One vertex of the rectangle is the centre of the circle and the opposite vertex is on the circumference, so you can use Pythagoras's Theorem to link x and y with the radius of the circle. This means that the equation you get will be quadratic, so you can expect to use the discriminant to determine whether it has any solutions.

In part (b) you need to show two things: that the value of P cannot be larger than the one you found, but also that there is a rectangle with this value of P.

In part (c), you could start again by writing equations connecting the sides and the perimeter of the second rectangle. However, if you split the circle into quarters, then each quarter has a rectangle inscribed in it in the same way as in part (b). You can therefore use the results you fond in part (b) about the largest possible value of the perimeter.

In the solution below we start by deriving the quadratic equation which will be used in all three parts.

Let x and y be the sides of the rectangle. The diagonal of the rectangle is a radius of the circle, so $x^2 + y^2 = 1$. The perimeter of the rectangle is $P = 2x + 2y$. Substituting $y = \dfrac{P - 2x}{2}$ from the second equation into the first gives

$$x^2 + \left(\frac{P - 2x}{2}\right)^2 = 1,$$

which is equivalent to

$$8x^2 - (4P)x + (P^2 - 4) = 0.$$

(a) When $P = 3$ this quadratic equation becomes $8x^2 - 12x + 5 = 0$. The discriminant is $12^2 - 4 \times 8 \times 5 = -16 < 0$ so there are no solutions for x. It is therefore not possible that $P = 3$.

(b) We now look for values of P for which the quadratic equation $8x^2 - (4P)x + (P^2 - 4) = 0$ has a solution. The discriminant needs to be non-negative, so we need

$$(4P)^2 - 32(P^2 - 4) \geq 0.$$

This is equivalent to $16P^2 \leq 128$ and, since $P > 0$, we must have $P \leq 2\sqrt{2}$. For this value of P, solving the quadratic equation gives $x = \frac{\sqrt{2}}{2}$ and, substituting back into $2x+2y = 2\sqrt{2}$, $y = \frac{\sqrt{2}}{2}$. Thus it is possible that $P = 2\sqrt{2}$ and this is the largest possible value of P.

(c) Let the sides of the rectangle be $2x$ and $2y$. The diagonal of the rectangle is a diameter of the circle, so $x^2 + y^2 = 1$. (Note that this is the same relationship between x and y as in part (b).) The perimeter of the rectangle is $4x + 4y = 2(2x + 2y)$, which is twice the perimeter from part (b).

We know from part (b) that the largest possible value of $2x + 2y$ is $2\sqrt{2}$, and that this value only occurs when $x = y = \frac{\sqrt{2}}{2}$. Therefore the largest possible value of our perimeter is $4\sqrt{2}$ and it occurs when the sides of the rectangle are equal, i.e. when it is a square.

5. Let n be a positive integer. Tracy writes a list of 10 whole numbers between 1 and n (inclusive). Each number in the list is either equal to, one less than, or one more than the number before it.

For example, when $n = 7$:

Her list could be 5, 5, 6, 7, 6, 6, 5, 5, 6, 6 or 4, 4, 3, 2, 1, 1, 1, 1, 1, 1.

Her list could *not* be 1, 3, 3, 4, 5, 5, 6, 7, 7, 7 or 5, 6, 7, 8, 7, 6, 5, 5, 5, 5.

(a) Suppose that $n = 3$. Stacey forms a list by copying Tracy's list, except that whenever Tracy writes a 1, Stacey writes a 3, and whenever Tracy writes a 3, Stacey writes a 1.

(i) Which lists could Tracy write that would cause her list to be the same as Stacey's?

(ii) Explain why Tracy can write as many lists that start 2, 2, 1 as start 2, 2, 3.

(3 marks)

(b) For which n between 1 and 10 (inclusive) is the number of lists that Tracy could write odd? (7 marks)

Solution

Commentary

There seems to be a lot going on in this question, so it is probably a good idea to start by writing out some lists. First make sure that you understand the rules. For $n = 3$, can you write some of Tracy's lists and the corresponding Stacey's lists? What makes the lists the same? What makes them different?

Part (a)(ii) asks you to consider lists of two different types (those starting 2, 2, 1 and those starting 2, 2, 3) and to show that there is the same number of each type. A useful way of showing that two sets contain the same number of elements is to try and pair them up. Can you see how Stacey can help you? You may well find that thinking in terms of Tracy's and Stacey's lists is the easiest way to write an explanation.

In part (b), you will need to adapt Stacey's rule slightly. Don't forget to check and explain why Stacey's new rule always generates a valid list (where each number is either equal to, one less or one more than the number before it). If the total number of Tracy's lists is odd, what does that tell you about pairing them up with Stacey's lists?

(a) (i) If Tracy writes a 3 at any point in her list, then Stacey will write a 1 at that point and so the lists will be different. Also, is Tracy writes a 1 at any point in her list, Stacey will write a 3 at that point and so the lists will be different. So, the only list that Tracy could possibly write that would cause Stacey to write the same list is the list where all ten entries are 2s. Indeed, this list does cause Stacey to have the same list as Tracy.

(ii) Note that whenever Tracy writes down a valid list of numbers, the list Stacey writes down is also a valid list of numbers. For any list L that Tracy can write, let $S(L)$ be the list that this causes Stacey to write down.

Suppose Tracy writes a list L that starts 2,2,1. Then $S(L)$ will start 2,2,3. We can get back to L from $S(L)$ by replacing all of the 1s in $S(L)$ with 3s and all of the 3s in $S(L)$ with 1s. In symbols, this says:

$$S(S(L)) = L.$$

So we can pair up each list L that Tracy could write beginning 2,2,1 with the list $S(L)$ beginning 2,2,3 that Stacey writes. Since this gives every list beginning 2,2,1 a unique partner beginning 2,2,3 (and vice versa), there must be the same number of lists beginning 2,2,1 as begin 2,2,3.

(b) Suppose for each n that whenever Tracy writes a list, Stacey copies Tracy's list except that whenever Tracy writes k, Stacey writes $n + 1 - k$. If Tracy writes a list L, write $S(L)$ again for the list that this causes Stacey to write.

Suppose Tracy writes a list which causes Stacey to write down the same list, i.e. $L = S(L)$. If an entry in Tracy's list is k, then we must have $k = n + 1 - k$, so $k = \frac{n+1}{2}$. Therefore, if n is odd there is one list where Stacey and Tracy write down the same list, namely

$$\frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}.$$

If n is even, then $\frac{n+1}{2}$ is not a whole number so there are no lists that cause Stacey and Tracy to write down the same list.

Now, if Tracy writes the list L and this causes Stacey to write the list $S(L)$, we can recover L from $S(L)$ by replacing each entry of $n + 1 - k$ with k, for $1 \leq k \leq n$. Since $n + 1 - (n + 1 - k) = k$, this is the same as doing Stacey's operation a second time, so

$$S(S(L)) = L.$$

So, all of the possible lists can be broken up into pairs of lists, where we pair the list L with its partner $S(L)$. If no lists are paired with themselves, this splits all of the possible lists into pairs, so there must be an even number of possible lists. Therefore, in the case n is even, there is an even number of possible lists. In the case n is odd, there is one list M that is paired with itself. So, the number of possible lists except for M is even, and therefore the total number of possible lists including M is one more than an even number. So, the total number of possible lists is odd whenever n is odd.

The possible values of n for which Stacey can write an odd number of lists are 1, 3, 5, 7 and 9.