United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. Each solution is preceded by a commentary, which suggests some ideas that might be useful in finding a solution.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Although in this year's paper you were required to give answers only, it may be useful to think about how you would justify your solutions. A full justification should explain why your method solves a problem and, if relevant, why there are no other solutions. You may want to try writing full solutions before reading on.

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1. (a) Let p and q be different prime numbers, and let $a = p^2$, $b = pq$ and $c = p^3q$. Which of ab, bc and ca is a square number? (1 mark)

Halle would like to place nine of the ten integers $1, 2, \ldots, 9, 10$ into the cells of a 3×3 grid in such a way that the three numbers within each row are written in increasing order from left to right and the product of the three numbers within each row is equal to a square number.

(b) Give an example showing that Halle's task is possible. (4 marks)

(c) How many different grids can Halle produce? (5 marks)

SOLUTION

COMMENTARY

This question is about multiplying integers to make square numbers. The aim of part (a) is to get you thinking how prime factors combine to make squares. It hopefully leads you to realise that, in a square number, each prime factor must appear an even number of times. This should help you decide which nine of the ten numbers from 1 to 10 Halle should use, and also which can combine together to make squares.

This illustrates a very important result about prime numbers: If p and q are (different) prime numbers, and if $p^m q^n$ is a square, then both p^m and q^n must be squares. Notice that this does not apply if p and/or q is not a prime; for example, $6^3 \times 24^1 = 5184 = 72^2$, but neither 6^3 nor 24^1 are themselves squares. This is just one example of why prime numbers are so special and important in mathematics.

Part (b) asks you to find only one example. While looking for the example, you should make note of any choices you can make; this will help you count how many different grids Halle can produce in part (c).

You should remember that, although the numbers in each row need to be increasing, the rows themselves can we swapped round. So for each way of grouping the numbers into three groups of three, there are several different grids, because the rows can be swapped round. You can count how many different arrangements of the rows there are by thinking about how many options there are for the first row, how many for the second, and how many for the third.

With only three rows, you can also label them A, B and C and write out all possible arrangements. With more rows, the listing becomes difficult. You may know about the *factorial function* which counts the number of possible arrangements of n objects: $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ — can you explain why this is?

(a) You can see that $ab = p^3q$, $bc = p^4q^2$ and $ca = p^5q$. Of these, $bc = (p^2q)^2$, so bc is the square number.

(b) In a square number, each prime factor must appear an even number of times. So we need to find groups of three numbers so that, in each group, each prime factor appears an even number of times.

First we note that 7 only appears once in the ten numbers, so it can not be used. At this point, it is useful to write down the prime factorisations of the remaining nine numbers:

$$1, 2, 3, 4 = 2^2, 5, 6 = 2 \times 3, 8 = 2^3, 9 = 3^2, 10 = 2 \times 5.$$

Next, 5 only appears as a factor in 5 and 10, once in each, so those two must go together. Since $5 \times 10 = 50 = 5^2 \times 2$, the third number in this group must have the factor of 2 appearing either once or three times. Any other prime factors would need to appear to an even power. Looking at the list above, the only numbers that can go with 5 and 10 are 2 or 8.

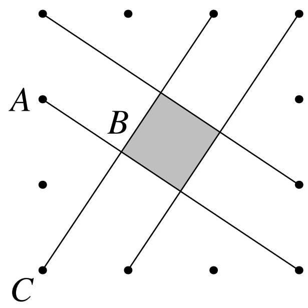
Suppose Halle chooses 2, 5 and 10 as the first group. Then 8 must go with 6 to make an even power of 2. But this group must also contain 3 to make an even power of 3. The final group is then 1, 4 and 9.

You should check that all three groups give square numbers: $2\times5\times10 = 100$, $3\times6\times8 = 144$ and $1 \times 4 \times 9 = 36$.

(c) When constructing the example in part (b), we had one choice to make: whether 2 or 8 went with 5 and 10. Choosing 2 gave us only one possible way to group the remaining numbers. If we choose 8 instead, then 2 must go with 6 and 3 must go with them as well, for the same reasons as given before. This gives another possible grouping: $5 \times 8 \times 10 = 400$; $2 \times 3 \times 6 = 36$; $1 \times 4 \times 9 = 36$.

For each of the two possible groupings, the three rows can be arranged in $3 \times 2 \times 1 = 6$ ways, so the total number of possible grids Halle can produce is $6 \times 2 = 12$.

2. Twelve points, four of which are vertices, lie on the perimeter of a square. The distance between adjacent points is one unit. Some of the points have been connected by straight lines. B is the intersection of two of those lines, as shown in the diagram.



(a) Find the ratio $AB : BC$. Give your answer in its simplest form. (3 marks)

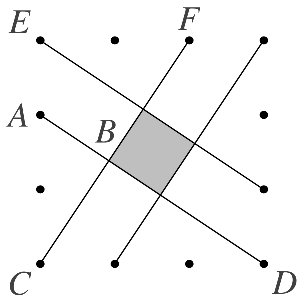
(b) Find the area of the shaded region. (7 marks)

Solution

Commentary

There are various ways to approach this problem; we will give two solutions using similar triangles and another using coordinate geometry.

You are first asked to find the ratio of two lengths, suggesting that you might want to use similar triangles. Consider the angles of triangle ABC. After labelling some additional points as shown in the diagram, you can see that angle CAB is shared with triangle DAC and angle ACB is shared with triangle FCE. Using this observation, what similar triangles can you find?



It seems "obvious, by symmetry" that the shaded region is a square, but such assertions should normally be proved. In particular, the symmetry of the diagram suggests that the shaded region has four equal sides, but it is less clear that it also has right angles. Our solution will include a proof of this fact.

If we call the length of the side of the shaded square x, then $AB + x + BC = \sqrt{3^2 + 2^2} = \sqrt{13}$.You know the ratio $AB : BC$ from part (a), and triangle ABC gives you another piece of information about the two lengths: $AB^2 + BC^2 = 2^2 = 4$. (Note that the analysis in part (a) implies that ABC is a right angle.) Can you see how you can use this?

It is also possible to find the area of the shaded region without calculating any lengths. To do this, note that the large square (which has area 9) is made up of four copies of triangle ABC and four copies of the quadrilateral region. Can you use the fact that triangles ABC and ACD are similar to find the area of triangle ABC?

(a) Using the diagram above, $\angle CAB = \angle CAD$ and $\angle ACB = \angle ECF$. But triangles ACD and FEC are congruent, so $\angle ACB = \angle CDA$. It follows that triangle ABC is similar to triangle ACD, so the ratio of its sides is $AB : BC = 2 : 3$.

(b) From the similarity of triangles ABC and ACD it follows that ABC is a right angle. By symmetry, the shaded region is a square. It is therefore sufficient to find the length of one of its sides.

Triangle ABC is right angled and its hypotenuse has length 2 units. Hence $AB^2 + BC^2 = 4$. But AB and BC are in the ratio $2 : 3$, so their squares are in the ratio $4 : 9$. Since they add up to 4, it follows that $AB^2 = \frac{4}{13} \times 4 = \frac{16}{13}$ and $BC^2 = \frac{9}{13} \times 4 = \frac{36}{13}$. Hence $AB = \frac{4}{\sqrt{13}}$ and $BC = \frac{6}{\sqrt{13}}$.

Let x be the length of the side of the shaded square; then $AB + x + BC = \sqrt{2^2 + 3^2} = \sqrt{13}$. Therefore,

$$x = \sqrt{13} - \frac{4}{\sqrt{13}} - \frac{6}{\sqrt{13}} = \frac{13 - 4 - 6}{\sqrt{13}} = \frac{3}{\sqrt{13}},$$

and so the area of the shaded square is $\frac{9}{13}$.

Alternative

(b) We know from part (a) that triangle ABC is similar to triangle ACD. The smaller triangle has hypotenuse of length 2 and the larger one has hypotenuse of length $\sqrt{13}$, so the scale factor of the similarity is $\frac{2}{\sqrt{13}}$.This means that the area scale factor is $\frac{4}{13}$. Since triangle ACD has area 3, it follows that triangle ABC has area $3 \times \frac{4}{13} = \frac{12}{13}$.

Triangle ACD is made up of two copies of triangle ABC and a quadrilateral region. Hence the area of the quadrilateral region is $3 - 2 \times \frac{12}{13} = \frac{15}{13}$. Finally, the large square, which has area 9, is made up of the required shaded region, four copies of triangle ABC, and four copies of the quadrilateral region. The area of the shaded region is therefore

$$9 - 4 \times \frac{12}{13} - 4 \times \frac{15}{13} = \frac{9}{13}.$$

Alternative

We use coordinate geometry to answer both parts. Let the origin be at C so that A has coordinates $(0, 2)$ and D has coordinates $(3, 0)$. Then the equation of the line CB is $y = \frac{3}{2}x$ and the equation of the line AB is $-\frac{2}{3}x + 2$. This shows that the two lines are perpendicular and so, by symmetry, the shaded region is a square. B is the intersection of the two lines, so its coordinates are $(\frac{12}{13}, \frac{18}{13})$.

(a) We can now calculate the lengths $AB = \sqrt{(\frac{12}{13})^2 + (\frac{8}{13})^2} = \sqrt{\frac{208}{169}}$ and $BC = \sqrt{(\frac{12}{13})^2 + (\frac{18}{13})^2} = \sqrt{\frac{468}{169}}$. Their ratio is $AB : BC = \sqrt{208} : \sqrt{468} = \sqrt{4} : \sqrt{9} = 2 : 3$.

(b) We only need to find one more vertex of the shaded square. The line through E parallel to AB has equation $y = -\frac{2}{3}x + 3$, and intersecting this with the line BC gives another vertex with coordinates $(\frac{18}{13}, \frac{27}{13})$. If we write x for the length of the side of the shaded square, we then have:

$$x^2 = \left(\frac{18}{13} - \frac{12}{13}\right)^2 + \left(\frac{27}{13} - \frac{18}{13}\right)^2 = \frac{36}{169} + \frac{81}{169} = \frac{117}{169} = \frac{9}{13}.$$

This is the area of the shaded square.

3. (a) The expression $10xy - 2x + 5y - 1$ factorises as $(ax + b)(cy + d)$ where a, b, c, d are integers and $a > 0$. Find the values of a, b, c, d. (1 mark)

(b) Using the factorisation above and considering factor pairs of 99, find all integer pairs (x, y) such that $10xy - 2x + 5y = 100$.
Enter y values *only* on the answer sheet. (4 marks)

(c) Find all integer pairs (x, y) such that $10xy - 2x + 5y = 100000001$.
Enter y values *only* on the answer sheet. (5 marks)

SOLUTION

COMMENTARY

You are probably used to factorising quadratic expressions such as $10x^2 + 3x - 1$. One method you may have been taught is to re-write the expression as $10x^2 - 2x + 5x - 1$ and then factorise in pairs. Can you see how to adapt this method to answer part (a)?

When you solve quadratic equations, you usually want one side of the equation to be zero. If $ab = 0$, then at least one of a and b must be zero. An equation like $ab = 99$ has infinitely many solutions: for any $a \neq 0$ you can take $b = \frac{99}{a}$. However, if you want your solutions to be integers, then there is only a finite number of possible pairs, corresponding to the factors of 99. This observation should help with both parts (b) and (c).

You will notice that some factor pairs do not lead to integer solutions. You should also remember that integers can be negative!

(a) We can factorise in pairs:

$$10xy - 2x + 5y - 1 = 2x(5y - 1) + (5y - 1) = (2x + 1)(5y - 1).$$

Hence $a = 2, b = 1, c = 5, d = -1$.

(b) We need to subtract 1 from both sides so we can use the factorisation from part (a). The equation is equivalent to $(2x + 1)(5y - 1) = 99$.

The factor pairs of 99 are 1×99, 3×33 and 9×11. These can be in either order, and can also be negative. This gives twelve different possibilities to be checked. We can cut out some of the work by noticing that $5y - 1$ is one less than a multiple of 5, which only leaves the options $5y - 1 = 99, 9, -1, -11$, with the corresponding $2x + 1 = 1, 11, -99, -9$.

Hence the solutions for (x, y) are $(0, 20)$, $(5, 2)$, $(-50, 0)$ and $(-5, -2)$.

(c) Subtracting 1 from both sides and factorising gives the equation $(2x + 1)(5y - 1) = 10^8$.

This has a lot of factors, all of the form $\pm 2^a \times 5^b$ where $0 \leq a, b \leq 8$. However, notice that $2x + 1$ is not a multiple of 2 and $5y - 1$ is not a multiple of 5.

So the only options are $2x + 1 = 5^8, 5y - 1 = 2^8$ and $2x + 1 = -5^8, 5y - 1 = -2^8$. The first of these gives $y = \frac{257}{2}$, which is not an integer, so the only integer solution is $x = -\frac{5^8+1}{2}, y = -\frac{255}{5} = -51$.

4. Daniel and Alessia each have eight boxes labelled 1 to 8. To start, each of them has n balloons, all in their box 1.

(a) Alessia is playing a game with her balloons. She may make the following move:

- Choose a box k containing at least two balloons. Pop a single balloon in box k and then move another balloon from box k to box $k + 1$.

Find the value of n such that, after a finite number of moves, all that remains is a single balloon in box 8. (2 marks)

(b) Daniel is playing a *different* game with his balloons. He may use any combination of the following two moves:

- Pop two balloons in box k and move a third balloon from box k to box $k + 1$.
- Pop a single balloon in each of boxes k and $k + 1$ and then move another balloon from box $k + 1$ to box $k + 2$.

What is the *smallest* value of n such that after a finite number of moves, all that remains

(i) is a single balloon in box 4?

(ii) is a single balloon in box 8? (8 marks)

Solution

Commentary

It is a good idea to try some examples to develop your understanding of the rules. For example, with Alessia's game, how many balloons do you need in box 1 in order to be able to get a balloon into box 2, or box 3? You may already be spotting a pattern, which you should try to explain. It may help to think "backwards", starting with one balloon in box 8 and asking how many were needed in box 7, then box 6, and so on.

With Daniel's game, the pattern is a bit harder to spot. You may quickly realise that it is more efficient to use the second move whenever you can, because it enables you to move a balloon two boxes forward. Can you write a more rigorous proof of this observation?

Thinking back from the final box is still a good idea. For example, to get a balloon in box 4 using the second move, you need two balloons in box 3 and another one in box 2. To get two balloons in box 3, you need four balloons in box 2 and another one in box 1. However, to get a balloon in box 2, you need to use the first move, which needs three balloons in box 1. You need to do one more step to complete the solution to part (b)(i).

If you want to use this method to work back from box 8, you need a systematic way to record the number of balloons needed at each "stage". Try completing this table:

8	7	6	5	4	3	2	1
1							
	2	1					
		4	$2 + 2 = 4$	1			

An alternative method is to come up with a recurrence relation. This is an equation that tells you how many balloons you need to get into box m if you know how many balloons you need to get into previous boxes. For example, the recurrence relation for part (a) is $a_m = 2a_{m-1}$: the number of balloons needed to get into box m is twice the number of balloons needed to get into box $m - 1$. In part (b), if you are using the second move then the number of balloons required to get into box m will depend on the numbers required to get into both boxes $m - 1$ and $m - 2$. You just need to work out how many balloons are needed to get into boxes 1 and 2, and then you can use the recurrence relation to find the number required to any subsequent box. We will use this method in our first solution for part (b).

(a) To get a balloon in a given box, Alessia needs two balloons in the previous box. Let b_m be the number of balloons required to get a single balloon into box m. Then $b_m = 2b_{m-1}$. Since $b_1 = 1$ (we only need one balloon in box 1), $b_8 = 2^7 = 128$.

(b) Let a_m be the least number of balloons that Daniel needs to have in box 1 at the beginning, to allow him (eventually) to get a single balloon into box m —so the question is asking us to find the values of a_4 and a_8 , but it turns out to be useful to find the values of a_m for other small values of m too.

Clearly $a_1 = 1$ and $a_2 = 3$.

Now suppose $m \geq 3$: as getting a balloon into box m can't be done without first getting a balloon into box $m - 1$, we see that $a_m \geq a_{m-1}$ (and $a_{m-1} \geq a_{m-2}$ which we'll use later).

Below we'll find the key recurrence relation $a_m = 2a_{m-1} + a_{m-2}$ (true for $m \geq 3$) and then use it to calculate the values of a_4 and a_8.

Each move Daniel makes consists of removing two balloons and moving a third. Equivalently we could think of this as removing three balloons and adding a new one, which we could call the 'successor' of each of these three balloons. Then at any fixed point in the future, each individual balloon which was in box 1 at the beginning will either still exist, or it will have been replaced by a successor, or it will have been replaced by a successor which will in turn have been replaced by its own successor, or . . . (and so on). So each balloon at the start has 'contributed' towards exactly one balloon which exists at any particular point in the future – it can't have contributed to more than one balloon existing at the same time.

To get a balloon into box m (where $m \geq 3$) we need three immediate 'predecessor' balloons – either three balloons in box $m - 1$, or two balloons in box $m - 1$ and one balloon in box $m - 2$. Any balloon which existed at the beginning (in box 1) can only have contributed to at most one of these three balloons, by the previous paragraph.

This means that to get a balloon into box m there will need to be at least $2a_{m-1}$ original balloons to allow two balloons to reach box $m - 1$ and at least a_{m-1} or a_{m-2} different original balloons contributing to the other 'predecessor' balloon (which is in box $m - 1$ or $m - 2$). But $a_{m-1} \geq a_{m-2}$ and so we need a total of at least $2a_{m-1} + a_{m-2}$ balloons at the start. And in fact, if you have $2a_{m-1} + a_{m-2}$ balloons at the start, then you can get all three predecessor balloons and so you can get a single balloon into box m.

So $a_m = 2a_{m-1} + a_{m-2}$ which is the key result in this question, and is the recurrence relation which will help us to find the final answers: this recurrence gives that $a_3 = 6 + 1 = 7$, and then $a_4 = 14 + 3 = 17$, $a_5 = 34 + 7 = 41$, $a_6 = 82 + 17 = 99$, $a_7 = 198 + 41 = 239$ and $a_8 = 478 + 99 = 577$.

Hence Daniel needs at least 17 balloons to get a single balloon into box 4, and at least 577 balloons to get a single balloon into box 8.

ALTERNATIVE

This is the solution for part (b)(ii). The answer for part (b)(ii) can be found by starting from box 4.

We can complete the table started in the commentary. To get one balloon in box 8, we need two balloons in box 7 and one balloon in box 6. This completes the first two lines of the table. To complete the third line, for each balloon in box m shown on the second line, we add two balloons to box $m - 1$ and one balloon to box $m - 2$. We continue this process, competing each line in turn, until we get to box 2. For each balloon in box 2, we need three balloons in box 1. The completed table is shown below.

8	7	6	5	4	3	2	1
1							
	2	1					
		4	2 + 2 = 4	1			
			8	4 + 8 = 12	4 + 2 = 6	1	
				16	8 + 24 = 32	12 + 12 = 24	6 + 3 = 9
					32	16 + 64 = 80	32 + 72 = 104
						64	32 + 240 = 272
							192

The total number of balloons required is the sum of all the numbers in box 1:

$$9 + 104 + 272 + 192 = 577.$$

We have not explained why this strategy gives the smallest possible number of balloons; for example, could using the first move somewhere in the middle (and not just after getting to box 2) give a smaller total? To complete the argument, we need to explain why it is best to use the second move whenever possible, as was done at the start of the first solution.

5. Freya creates a sequence with first term 1 and each subsequent term 5 more than the previous term. Hilary creates a different sequence with first term a and each subsequent term 3 less than the previous term. Both sequences are continued forever.

(a) There is at least one number that appears in both sequences. Let c be the smallest of those numbers. Find the possible values of c. (3 marks)

(b) Given that there are exactly 100 numbers which appear in both sequences, find the possible values of a.

Enter the following on the answer sheet:

(i) How many possible values of a are there?

(ii) The smallest possible value of a.

(iii) The largest possible value of a.

(7 marks)

SOLUTION

COMMENTARY

The best way to approach a problem like this is to play around and see if you make any interesting observations. Freya' sequence goes: $1, 6, 11, 16, 21, \ldots$. Hilary's sequence is decreasing, so it may be easiest to write it out "backwards" starting from a common term. For example, the smallest common term for the two sequences could be 1, in which case Hilary's (backwards) sequence would go $1, 4, 7, 10, 13, 16, 19, \ldots$. Can you see a pattern in the terms which overlap with Freya's sequence? Can you prove your observation?

Could 6 be the smallest common term? What about 11, or 16, or 21? How is this related to the observation you made above?

For part (b), you may want to start by creating a sequence that has only two or three common terms with Freya's (rather than 100). For example, if $a = 16$ then Hilary's sequence has exactly two terms in common with Freya's (1 and 16). If $a = 19$, there are still the same two common terms, and this is also the case if $a = 22, 25$ or 28. However, if $a = 31$ there are three common terms, $1, 16, 31$. Can you see how to use similar reasoning to create a sequence which has exactly 100 terms in common with Freya's? Do you have to start from 1 as the smallest common term?

(a) Any common term must appear in Freya's sequence. 1 can clearly appear in both sequences, in which case it is the smallest common term (as there are no terms smaller than 1 in Freya's sequence). If 6 appears in Hilary's sequence, then it continues $3, 0, -3, \ldots$ and none of these overlap with Freya's sequence. So 6 could be the smallest common term. Similarly, if 11 appears in Hilary's sequence, then it continues $8, 5, 2, -1, \ldots$ which again does not overlap with Freya's, so the smallest common term could be 11. However, if 16 appears in Hilary's sequence, then so does 1, so 16 cannot be the smallest common term.

In general, suppose x appears in both sequences. Then Freya's sequence contains $x-5, x-10, x-15$ and Hilary's sequence contains $x-3, x-6, x-9, x-12, x-15$. So $x-15$ also appears in both sequences. Moreover, there are no other terms between x and $x-15$ which appear in both sequences.

Therefore the smallest common term can be at most 15, and the possible values of c are $1, 6$ and 11.

(b) We have shown in part (a) that successive common terms differ by 15. Therefore the 100 common terms are $c+15k$, where $k=0,1,\ldots,99$ and $c=1,6$ or 11.

When $c=1$, the 100th common term is $1+15\times 99=1486$, and this could be the first term of Hillary's sequence. However, Hilary's sequence could also start from 1489, 1492, 1495 or 1498 and the largest common term would still be 1486. But if Hilary's sequence started from 1501, then this would be the 101st common term. So when $c=1$ there are five possible values of a.

Similarly, when $c=6$, the first term of Hilary's sequence could be $6+15\times 99=1491$, and also 1494, 1497, 1500 or 1503, but not 1506. Finally, when $c=11$, the first term of Hillary's sequence could be 1496, 1499, 1502, 1505 or 1508.

Therefore, there are 15 possible values of a, the smallest being 1486 and the largest 1508.