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MATHEMATICAL OLYMPIAD FOR GIRLS 2021

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Introduction

For many students, and possibly some teachers, this is the first experience of attempting a Maths Olympiad paper. It may therefore be useful to understand how these papers are marked, as students may be disappointed to receive a small number of marks for a problem they thought they had almost solved.

All Olympiad papers are marked using what the call the '0+/10−' principle. This means that the markers first read the whole write-up and decide whether the student has a viable strategy to solve the problem. It may be that there are some mistakes or small gaps in their reasoning, but if those could be relatively easily filled in then this response is marked in the '10−' regime, with usually up to three marks being taken away for gaps and mistakes. Common examples of small gaps are algebraic or arithmetical errors (provided they don't change the nature of the argument), missing one of several cases in a counting question, or lack of geometrical reasons when calculating angles.

If, on the other hand, the student has only started to explore the problem and has only made some useful observations, but does not have a strategy to generalise or prove them, then the script is marked in the '0+' regime. Up to three marks may be available for spotting a pattern or trying an idea which, if progressed further, could lead to a solution. The example of the former in the present paper would be, in Question 4, trying the game for $A = 1$ to 10 and conjecturing the correct answer. An examples of the latter would be using algebra to describe the tiles in Question 2, or introducing some variables for the unknown lengths in Question 3 and writing down the area and perimeter equations. Notice that all those examples involve a substantial engagement with the problem, rather than just trying one or two examples. Teachers should therefore reiterate to students that scoring even one or two marks on any of these questions is a real achievement.

It is unfortunately often the case that students think that they have solved a problem but only receive two or three marks. The most common reason for this is that their solution relies on a series of unjustified claims. The prime example in this paper was Question 4, where many candidates made correct claims about the number of odd numbers in various cases, or about the parity of the expression $\frac{A(A-1)}{2}$. If those claims were purely made on the basis of observation, with no justification of why they are true in general, then this does not constitute a proof and can only receive marks in the '0+' regime. Students in this situation are strongly advised to read these comments and the official solution, to understand how they can add sufficient detail to their proofs.

The Girls' Olympiad paper is slightly different from other Maths Olympiads in that questions are broken down into several parts. Most of the time, the final part is the "main question" and the first part (or parts) are intended to suggest some useful results or good approaches to the problem. The reason for structuring the paper in this way is that the setters know that many of the candidates are not experienced in olympiad mathematics, and the hope is that by giving these pointers, we enable them to engage with a question even if they are not familiar with some standard olympiad technique or "trick". A useful hint is to read the whole question first and try to understand how the early parts may be helpful in solving the main problem.

General comments

The markers were pleased to see so many good answers to the questions on this paper, including many good attempts at the very challenging Question 5. We were particularly impressed that on Questions 2 and 4 so many candidates engaged with the problem and made some insightful observations, and tried to explain them, even if their justifications were not complete. It is pleasing to see that such a large number of students understand the importance of explanation and proof.

We saw some excellent examples of mathematical communication with candidates choosing wisely between words, equations and diagrams. Olympiad paper markers usually comment that students should use more words to explain what they are doing, but in this paper we often observed the opposite: candidates were trying to explain their calculations in words where a combination of words and equations would have been clearer. One example of this was Question 2 part (a), where we often saw statements like 'The numbers on either side of the middle tile are the same distance away, so the total is three times the middle number' – this is true, but could be expressed more elegantly using algebra as $(x - 1) + x + (x + 1) = (x - 4) + x + (x + 4) = 3x$.

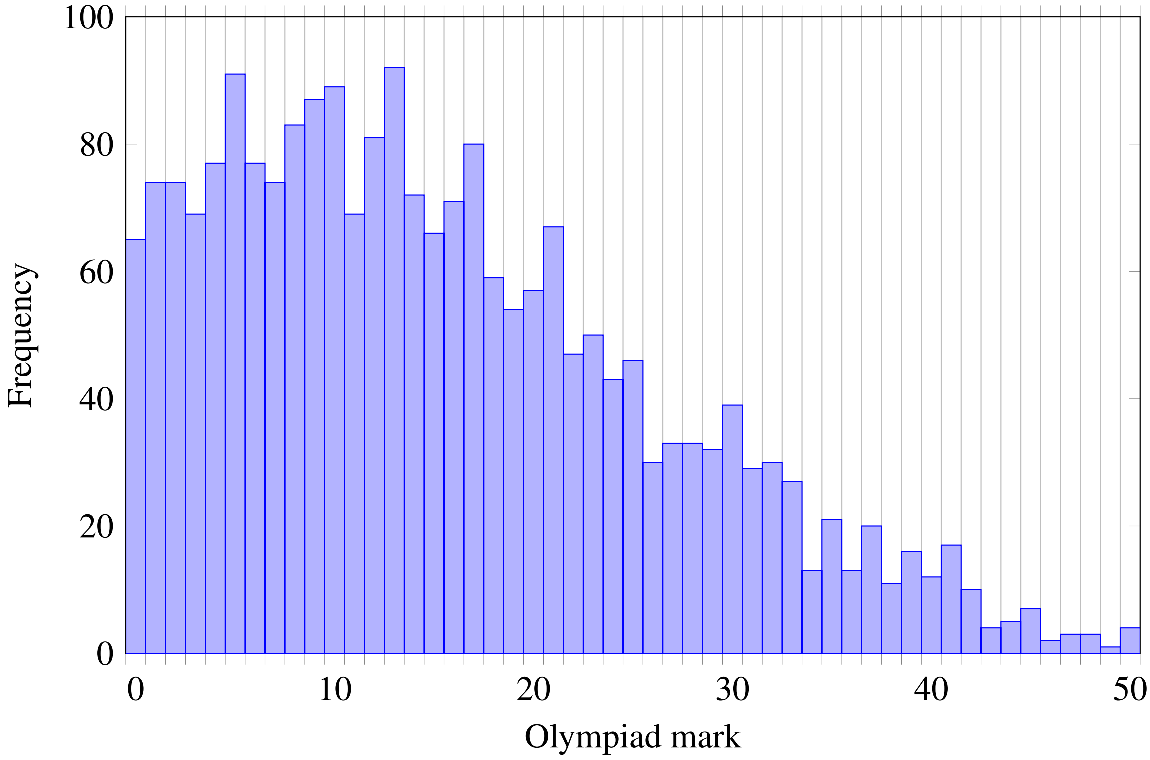
Sometimes the use of algebra was not quite precise enough, although it was clear what was being said. For example, in Question 4 we often saw statements like 'even + odd = odd' justified by writing $2n + (2n + 1) = 4n + 1$. This does not prove the statement in general, since 'n' for the two numbers need not be the same. A correct equation would be $2n + (2m + 1) = 2(n + m) + 1$. On the other hand, in Questions 1 and 3 we saw some confident and accurate manipulation of quadratic equations.

The paper revealed some common misconceptions which teachers could use for discussion with their classes. Most of these seem to stem from trying to "guess a rule" rather than using examples to explore it. Some of the common incorrect statements we saw are: 'One less than a multiple of 4 is a multiple of 3', 'The sum of an odd number of numbers is odd' and 'A non-multiple of 3 minus another non-multiple of 3 is a multiple of 3'. All of these statements can be explored using examples, hopefully leading to the discovery of the correct "rules". Stronger students can then be challenged to try and prove the correct statements, but even the less confident students can benefit from the idea that mathematical rules can be explored and discovered, rather than just learnt.

Mark distribution

The MOG 2021 paper was marked online by a team of Alan Slomson, Alfred London, Amit Goyal, Amit Shah, Amit Srivastava, Andjela Sarkovic, Andrew Ng, Daniel Claydon, David Vaccaro, Frankie Richards, Ina Hughes, Jeremy King, Jordan Baillie, Joseph Myers, Kasia Warburton, Kit Kilgour, Laura Daniels, Martin Orr, Mary Teresa Fyfe, Matthew Smith, Melissa Quail, Michael Illing, Michael Thornton, Naomi Bowler, Patricia King, Paul Druce, Paul Scarr, Peter Price, Phillip Beckett, Richard Freeland, Robin Bhattacharyya, Sam Bealing, Sam Neil, Sophie Maclean, Stefan Dixon, Stephen Tate, Sue Cubbon, Sylvia Neumann, Tom Bowler, Vesna Kadelburg, Vivian Pinto and Wendy Dersley.

We received non-empty scripts from 2229 candidates.



Question 1

(a) Find all whole numbers x such that

$$(x^2 - 7x + 11)^{(x^2-4x+4)} = 1.$$

(7 marks)

(b) Find all whole numbers x such that

$$(x^2 - 7x + 11)^{(x^2-4x+4)} = -1.$$

(3 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

For the first question on the paper the aim for the setters is to make it accessible to all candidates, and whilst pleasingly we saw over 40% of candidates achieving 7 marks or more we also saw that 25% of candidates achieved fewer than 3 marks. There were a number of common errors and oversights which meant that the modal mark was 7, with a quarter of candidates achieving this.

The most common oversight was to forget to check to see if the base could be (-1) and the power even in part (a). Remarkably many candidates correctly solved part (b) by finding out when the base was (-1) but failed to realise that they had in fact found another solution to part (a)!

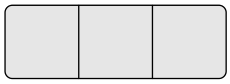
A relatively common error in both parts was to assume that the power had to be 1, which did lead to some correct solutions but is not true in general because any power of 1 gives 1. A less frequent error was that candidates incorrectly believed that $1^{-1} = -1$ leading to the false claim that there are no solution to part (b). For any candidate studying for their GCSE or A-Level in Mathematics it is vitally important they learn the rules of indices as they are fundamental to much of the algebra we do.

For a competition aimed at Year 11, 12 and 13 students we would expect solving quadratic equations to be a key skill all candidates had previously mastered. Unfortunately, it was clear that some candidates were unfamiliar with quadratic equations and how to solve them. There were often trial and error methods employed and correct solutions found, but usually these were not awarded marks because there was no reasoning provided as to why there were no more solutions. In a very small number of cases candidates successfully argued that the base and the power became too large when $x > 7$, but most arguments were vague at best and awarded very few marks if any. It is important that candidates remember that when attempting a Mathematical Olympiad problem, any claim they make that is relevant to their solution must be fully justified. This justification will often take the form of algebra as it did in this problem but this is not always the case.

Question 2

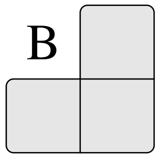
Consider a 4×4 grid numbered 1 to 16 left to right then top to bottom. Tile A or Tile B is placed onto the grid so that it covers three adjacent numbers.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16



A

B



(a) If Tile A is placed onto the grid (the orientation of the tile may be changed), can the total of the **uncovered** numbers be a multiple of three? (3 marks)

(b) In how many different ways can Tile B be placed onto the grid (the orientation of the tile may be changed) so that the sum of the **uncovered** numbers is a multiple of three? (7 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

This was both the most popular (83% of candidates scoring marks) and the highest scoring question; in fact, the modal mark was 10, although there was also a large number of low-scoring attempts. We were impressed by a variety of solutions and some excellent communication skills.

It is possible to answer this question by simply listing and checking all possible positions of the two tiles: there are only eight possible positions for Tile A and 36 for Tile B. Those studying A Level will know that this is a valid method, called proof by exhaustion, and it can score full marks. However, those attempting it should be careful to list all the possibilities in a systematic order, to ensure that none are repeated or left out, and any arithmetical errors will lead to the loss of most of (or even all) the marks.

Some candidates attempting the listing method spotted a pattern in the covered or uncovered totals, although very few were able to explain it fully. For example, for Tile B in the 'L' orientation, the uncovered totals are 124, 121, 118, 112, 109, ... This observation forms a basis of a short and elegant method, as follows. When a tile is moved horizontally, each square's value changes by 1, so the total covered (or uncovered) changes by 3. When a tile is moved vertically, each square changes by 4 so the total changes by 12. This means that, for a given orientation of a tile, either all possible position will give a multiple of three, or none of them will. Hence we only need to check the uncovered totals when the tile is in the top-left corner. For Tile A these are $136 - (1 + 2 + 3) = 130$ and $136 - (1 + 5 + 9) = 121$ and for Tile B they are $136 - (1 + 2 + 5) = 128$, $136 - (1 + 2 + 6) = 127$, $136 - (1 + 5 + 6) = 124$ and

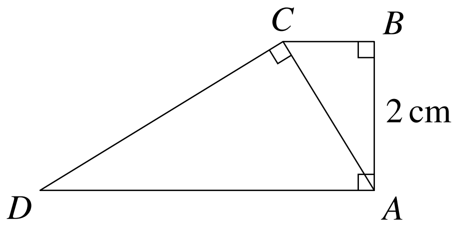
$136 - (2 + 5 + 6) = 123$. Of these, only the last one is a multiple of 3, so only one orientation of Tile B works.

The most commonly successful method was the one presented in the official solution, using algebra to find expressions for the covered and uncovered totals. Any algebraic solution should start by defining variables clearly, and most candidates did this by drawing and labelling the tiles.

Another elegant solution involves replacing the numbers in the grid by the remainders they give when they are divided by 3. More experienced students will know that this is called "arithmetic modulo 3", but we were really impressed by the number of candidates who clearly came up with the idea by themselves, sometimes using colours or shapes to represent different remainders. Once you have formed the grid of remainders (which begins 1, 2, 0, 1; 2, 0 . . .) it becomes clear that Tile A covers one of each number, so the total covered is always $0 + 1 + 2$ which is a multiple of 3. For Tile B there are more options, but we only need to look for ones that work. Since 136 gives remainder 1 when divided by 3, to get a multiple of 3 the covered total also needs to leave remainder 1. This means that the three covered numbers must add up to 1 or 4. The only ways to get one of these are $1 + 0 + 0$, $1 + 1 + 2$ and $2 + 2 + 0$. Each can be found in the grid three times, corresponding to nine possible placements of Tile B.

Question 3

The diagram shows a quadrilateral $ABCD$, where AB is 2 cm and $\angle ABC$, $\angle ACD$ and $\angle DAB$ are right angles.



(a) Let E be the point on DA such that CE is perpendicular to DA. Prove that triangles ABC and DEC are similar. (2 marks)

(b) Given that the area of quadrilateral $ABCD$ is 6 cm^2, find all possible values for the perimeter of quadrilateral $ABCD$. (8 marks)

Solution

(See the official solutions document.)

Markers' comments

Geometry tends to be the least popular topic in Olympiads, but this question was answered well with 75% of candidates scoring marks and over a quarter essentially solving the problem. It was encouraging to see students finding several different methods to approach it.

In part (a), showing that the two triangles are similar required proving that angles in each triangle are equal, for example angles ACB and DCE (and also the right angles). A sequence of true equations involving angles was desired, but also some geometrical justification, for example 'angles in a triangle' or 'alternate angles' stated each time that those results were applied, and it needed to be observed (if used) that angle ECB is a right angle (this follows, for example, from angles in the quadrilateral $ABCE$). Many students lost a mark in part (a) for a lack of geometrical justification. The 2cm sides were not at all relevant to showing that these triangles are similar. In fact, looking at side lengths was not the way to go in part (a), so students trying such an approach were unsuccessful.

Some students will have thought that they had solved part (a) but actually hadn't – for two main reasons. Firstly, some students showed that triangles ABC and CEA are similar (in fact congruent) and stopped there – but it was necessary to look at angles in the triangle DEC to receive credit here. Others showed that triangles ABC and DCA are similar. Secondly, some students tried to use scale factors together with Pythagoras, observing that if two sides of triangle ABC were scaled up by the same scale factor then Pythagoras would be consistent with the third side being scaled up by the same scale factor. But this approach assumes from the start that the triangles are similar – so it's not a valid way of proving similarity. There were also some students who mistakenly assumed from the start that some angles had to be 45 degrees (they didn't have to be).

In part (b), the idea was to use part (a) as a hint – as is often the case in MOG questions. The similarity of triangles would give a relationship between BC and DE, and area considerations would give us another equation; then algebra would be required to solve these simultaneous equations – leading to a quadratic equation to be solved.

The major misconception in part (b) was assuming that BC or DE had to be integers. The area equation gave $DE + 2BC = 6$. Many students noticed that the only positive integer solutions to this equation are $BC = 1$ and $BC = 2$, and then they calculated the perimeters from there. This did in fact produce the correct two numerical answers for the perimeter, but this was really just a coincidence, and these students received little credit. It was necessary to eliminate all the non-integer possible solutions (for BC) and, for this, the similarity of triangles was also to be used.

There were some excellent alternative solutions found by students taking the paper. Part (b) could be solved without using part (a). For example, Pythagoras could be used in triangles ABC, DEC and DCA to discover that the product of DE and BC is equal to 4, and this equation can then be solved simultaneously with the equation coming from considering area.

A few students even went back to prove similarity in (a) after reaching this point in (b), using scale factors between the sides (deriving $\frac{DE}{2} = \frac{2}{BC}$ and concluding similarity).

There were other algebraic approaches for (b) that were found by some students, involving a mixture of Pythagoras and similarity of triangles (especially when using hypotenuses of similar triangles). One approach which worked involved finding the area of triangle DCA in two different ways, as DA and as $\frac{CA \times CD}{2}$ (with all these distances given in terms of BC and DE – using Pythagoras to express each of CA and CD in terms of these quantities). There was some excellent algebraic manipulation from many candidates in solutions of these types.

Trigonometry was also a successful method used by a few to solve part (b), especially if every relevant length could be put in terms of $\tan(BAC)$ or $\tan(ACB)$; this led to a quadratic equation. One student proved part (a) by using trigonometry, finding two scale factors as $\tan Z$ and as $\frac{1}{\sin Z \cos Z} - \frac{1}{\tan Z}$, and then proving that these are equal to each other, using a trigonometric identity.

One nice way to relate angles in triangles ABC and DEC (in part (a)) was to consider angles in the quadrilateral $ABCD$ – there were three right angles and also angles ACB and CDE, which must therefore sum to give 90 degrees. Finally, some students noticed that simply rotating triangle ABC by 90 degrees would give a triangle with all its sides parallel to the sides of triangle DEC – leading to equal angles in the triangles.

Question 4

Sam is playing a game. Her teacher gives her a positive whole number A, and then Sam chooses a positive whole number S. Sam then adds together all of the integers between S and $S + A - 1$ (inclusive) to obtain a total T. If T is even, Sam wins the game. For example, if $A = 4$, Sam can win by choosing $S = 10$ because then $T = 10 + 11 + 12 + 13 = 46$.

(a) (i) Show that if $A = 4$, Sam will win the game no matter which number she chooses.

(ii) Show that if A is a multiple of 4, Sam will win the game no matter which number she chooses. (3 marks)

(b) For which other values of A can Sam choose an S so that she wins? You must show how she can win for each of those values, and also explain why she cannot win for all the other values. (7 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

We were really impressed with the level of engagement with this question, with 80% of candidates scoring some marks on it, and just under a quarter making substantial progress. Many more candidates found the correct answer, often by trying a few examples and spotting a pattern, but were not able to justify why the pattern continues beyond their examples. Explaining how we can be sure that patterns continue forever is a major aspect of mathematical proof, and such candidates will really benefit from reading these notes carefully.

For the first part, there was about an even split between candidates using algebra ($S + (S + 1) + (S + 2) + (S + 3) = 4S + 6$) and those noticing that four consecutive numbers must contain two evens and two odds, which always sum to an even number.

For the second part of (a), there were many successful uses of the formula for the sum of consecutive numbers. Those trying to link part (ii) to part (i) often had the right idea of considering blocks of 4, but many made incorrect statements in their explanations. For example, a common claim was 'When $A = 4$, $T = 4S + 6$, so when $A = 4n$ we will have $T = n(4S + 6)$.' This is trying to use algebra to express the idea that n blocks of $4S + 6$ are being added together, but those blocks are not all the same; a correct statement would be something like $T = (4S_1 + 6) + (4S_2 + 6) + \dots$ It is worth noting that some ideas are more easily expressed using words than algebra, and good mathematical communication involves choosing the best representation. For example, the idea above is most clearly expressed simply by saying 'When $A = 4n$, there are n blocks of four added together, and each block of four has an even sum.'

Another common incorrect statement was that 'there is an even number of odd numbers because A is even'. This is clearly not true – for example, when $A = 6$, there are three even and three odd numbers. The correct statement is that, since A is a multiple of 4, half of A is also even, so the number of odd numbers is even.

Trying to count odd numbers was a popular approach to part (b) as well. This can be made into a rigorous argument, but unsubstantiated claims based of spotting a pattern did not receive many marks. For example, a common (correct) claim was that 'If A is a multiple of two but not a multiple of four, there is an odd number of odd numbers.' This can be justified relatively simply by saying that in this case, half of A is odd – notice how this clearly distinguishes this case from the case $A = 4k$ above. Counting odd numbers when A is odd requires slightly more thought, but we could say something like this: If $A = 2n + 1$ and we start on an even number, then there will be $n + 1$ even numbers and n odd numbers; so Sam can win if n (which is $\frac{A-1}{2}$) is even and S is even; a similar argument can be made about starting on an odd number.

As well as unjustified correct claims about even and odd numbers, we also saw many incorrect claims, which could have been easily checked by trying some examples. For example, 'A sum of an odd number of consecutive numbers is always odd' is clearly false, as can be seen from $1 + 2 + 3 = 6$.

Most successful candidates tackled this question by finding a formula for T, either in the form $T = \frac{A(2S+A-1)}{2}$ or $T = AS + \frac{A(A-1)}{2}$. Those using the first formula usually found that two of the possibilities are $A = 4k$ (already considered in part (a)) and $2S + A - 1 = 4k$, leading to $A = 4k - 2S + 1$, an odd number. In this case it needs to be justified that, for every odd A, it is possible to find an S such that $2S + A - 1$ is a multiple of 4. Trying to prove this leads to realising that the winning choice of S depends on the remainder when A is divided by 4. However, there is a third possibility, namely that neither A nor $2S + A - 1$ are multiples of 4 but both of them are multiples of 2. Although this proves to be impossible (because if A is even then $2S + A - 1$ must be odd), many candidates lost marks for failing to mention it at all.

The second version of the formula, $T = AS + \frac{A(A-1)}{2}$, leads to a similar analysis, although it is a little harder to classify all the cases. For the sum to be even, the two terms must be either both even or both odd. There is only one way for AS to be odd – when both A and S are odd numbers. But this doesn't mean that all odd numbers A work, because $\frac{A(A-1)}{2}$ must also be odd. Some experimenting suggests that this is the case for every other odd number, but this needs to be proved. The easiest way to do this is to write $A = 2k + 1$ in which case $\frac{A(A-1)}{2} = k(2k + 1)$ and this is odd when k is odd (so we can write $k = 2n + 1$). Hence S odd only works with $A = 4n + 3$. When both AS and $\frac{A(A-1)}{2}$ are even, there are two options: A even, or A odd and S even. We can then ask when $\frac{A(A-11)}{2}$ is even and find that the winning combinations are A being a multiple of 4 or $A = 4n + 1$ and S even. Candidates using this version of the formula often had a good idea of the answer, but were not able to classify all the cases clearly.

It should be noted that the question didn't explicitly ask for a choice of S for each value of A, but only for which values of A a winning S can be found. Some candidates exploited this successfully by using the following argument with the second formula, $T = AS + \frac{A(A-1)}{2}$. If A is odd, Sam can make AS the same parity as $\frac{A(A-1)}{2}$ by choosing S appropriately, so she can win for all odd A. For even A, she can only win if $\frac{A(A-1)}{2}$ is also even, in which case she will win with any choice of S. By writing $A = 2k$, we find that this only happens when A is a multiple of 4.

Finally, while we have pointed out at the start that answers found by pattern-spotting did not receive many marks, it is a useful starting point which allows you to formulate some correct statements (which then of course you need to prove) and, just as importantly, avoid making false ones.

Question 5

(a) By considering their difference, or otherwise, find all possibilities for the common factors of n and $n + 3$. (1 mark)

For $n \geq 2$, let $P(n)$ denote the largest prime factor of n.

(b) If a and b are positive integers greater than 1, explain why $P(ab)$ must be equal to at least one of $P(a)$ or $P(b)$. (1 mark)

(c) Find all positive integers n such that $P(n^2 + 2n + 1) = P(n^2 + 9n + 14)$. (8 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

This was a genuinely challenging problem, with only around 30 candidates essentially solving it and another 60 making significant progress. On the other hand, over 40% of candidates scored at least one mark, showing a willingness to engage with even the most difficult problems – they should be really proud of their attitude.

Many candidates attempted part (a) but did not fully understand what was required in order to get the mark: either they wrote down 1 and 3 as the possible common factors, but did not explain why no other common factors are possible; or they explained correctly why no other common factors are possible, but did not include 1 as a common factor.

The number of correct answers to part (b) was encouraging as it required understanding both the definition of $P(n)$ and the behaviour of prime factorisation. Unfortunately some candidates did not understand the function notation and interpreted $P(n)$ either as meaning "n is prime" or as something to do with probability.

Part (c) was a hard question whose solution required several steps, including a solid understanding of both parts (a) and (b). Many candidates factorised the quadratics, which is an essential first step, but only a few recognised that they could then use (b). Those who did often managed to get as far as $P(n + 1) = P(n + 7) = 2$ or 3. Making further progress required a careful analysis of possible cases and also ensuring that $P(n + 2)$ is not greater than the other two.

The 60 or so candidates who made substantial progress largely got stuck in the same place. The three possibilities for $n + 1$ and $n + 7$ are: they are both powers of 2, they are both powers of 3, or they have both 2 and 3 as factors (so that $n + 1 = 2^a 3^b$ and $n + 7 = 2^c 3^d$). The first two cases can be dealt with easily, as gaps between successive powers of 2 or 3 grow exponentially, so only the first few cases need to be checked. Unfortunately, the same is not true in the third case, as can be seen by looking at the sequence: $6, 12, 18, 24, 36, 48, 54, 72, 96, 108\dots$ So some more work is needed to narrow down the options.

One way to proceed is to cancel a factor of 6 to find that either $2^{c-1} - 3^{b-1} = 1$ or $3^{d-1} - 2^{a-1} = 1$ or $2^{c-1}3^{d-1} - 1 = 1$. The third equation clearly has only one solution ($a = 2, d = 1$). The first two are examples of the so-called Catalan Conjecture, a result that has only been proved recently, that says that the only consecutive powers of 2 and 3 are 8 and 9. This known fact could be quoted without proof.

It sounds unfair that a knowledge of such an obscure theorem is required to solve this problem. But the whole analysis of the last two paragraphs can be avoided by noticing that, if $n + 1$ and $n + 7$ have both 2 and 3 as factors, then $n + 2$ must have a factor that is not 2 or 3, which would make it larger than $P(n + 7)$, and therefore none of these options lead to a solution for n. Those who tried pairs such as 6 and 12 or 48 and 54 may have noticed this anyway.

This brings us to the final point: Just because we found an n that satisfies the condition $P(n + 1) = P(n + 7) = 2$ or 3, it does not mean that this is a solution to the original problem, as we also need $P(n + 2) \leq P(n + 7)$. It was therefore necessary, for the final mark, to check that $n = 2$ does indeed work.