United Kingdom
Mathematics Trust

# Mathematical Olympiad for Girls



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## Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. They are not intended to be the 'best' possible solutions; in some cases we have suggested alternatives, but readers may come up with other equally good ideas.

All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or been unclear in your explanation, then you will normally receive nearly all the marks.

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**1.** (a) Find all whole numbers $x$ such that

$$(x^2 - 7x + 11)^{(x^2-4x+4)} = 1.$$

(7 marks)

(b) Find all whole numbers $x$ such that

$$(x^2 - 7x + 11)^{(x^2-4x+4)} = -1.$$

(3 marks)

## Solution

### Commentary

This question looks intimidating at first, and you may think that some complicated algebraic manipulation is required. However, with a little thought you should realise that all you are being asked is: how can a number raised to a power equal 1?

For part (a), there are three options. Part (b) should make you think of the third one, in case you thought of just two options at first.

If you know about logarithms, it may be tempting to try and use them, as the question involves unknown powers. Unfortunately, because you cannot take a logarithm of a negative number, this does not help with part (b), or the related case from part (a).

(a) There are three possibilities for $a^b = 1$: Either $b = 0$, or $a = 1$, or $a = -1$ with even $b$. Each of these possibilities leads to a quadratic equation.

If $x^2 - 4x + 4 = 0$ then $x = 2$. (You may know that $0^0$ is undetermined, but you can check that when $x = 2$, the base is not 0, so there is no problem here.)

If $x^2 - 7x + 11 = 1$ then $x^2 - 7x + 10 = 0$ so $x = 2$ or 5.

If $x^2 - 7x + 11 = -1$ then $x^2 - 7x + 12 = 0$ which has solutions $x = 3$ and $x = 4$. Out of those, only $x = 4$ makes $x^2 - 4x + 4$ an even number, so it is the only solution in this case.

Therefore the solutions are $x = 2, 4, 5$.

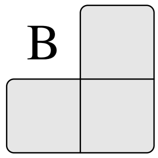
(b) We now need $x^2 - 7x + 11 = -1$ and $x^2 - 4x + 4$ odd. From part (a) we already have the solution in this case: $x = 3$.

**2.** Consider a $4 \times 4$ grid numbered 1 to 16 left to right then top to bottom. Tile A or Tile B is placed onto the grid so that it covers three adjacent numbers.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

A

B



(a) If Tile A is placed onto the grid (the orientation of the tile may be changed), can the total of the **uncovered** numbers be a multiple of three? (3 marks)

(b) In how many different ways can Tile B be placed onto the grid (the orientation of the tile may be changed) so that the sum of the **uncovered** numbers is a multiple of three?

(7 marks)

SOLUTION

COMMENTARY

It would be feasible on a grid this size to consider every possible placement of Tile A and Tile B and calculate the relevant totals of the **uncovered** numbers in each case. This would be tedious, time consuming and likely lead to careless arithmetic errors.

A more elegant approach would be to consider general positions for each of the tiles and try to find a relationship between the covered numbers. This can be easily achieved by introducing algebra in to the problem, as the diagrams below illustrate. In a problem of this kind it is likely that multiple positions of a given tile yield the same end result and if we can deal with many positions at once this is a huge advantage.

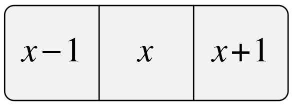
Another benefit of an algebraic method is that we could make minor adaptations to our argument to deal with larger grids or even different shaped grids.

(a) To begin we calculate the total of all sixteen numbers in the grid, which you may recognise as the 16th triangular number and can be calculated using the formula $T_n = \frac{1}{2}n(n+1)$. The total is 136, which is one more than a multiple of three.

Tile A can be placed horizontally or vertically.

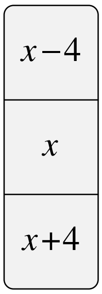
 

If we place it on to the grid in the horizontal position then the three numbers covered by the tile can be written as $x - 1$, $x$ and $x + 1$ for some $x$ between 1 and 16, as the diagram here shows.




We should note that not every value of $x$ in this range yields a valid position on the grid. The total of the three covered numbers is $3x$, which is always a multiple of three. This means that when Tile A is placed horizontally the total of the **uncovered** numbers is $136 - 3x$, which is one more than a multiple of three no matter the value of $x$.

If Tile A is placed vertically on to the grid then the three covered numbers can be written as $x - 4$, $x$ and $x + 4$, as the diagram here shows.

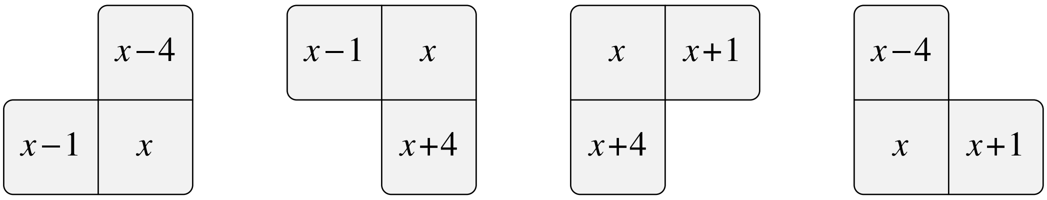



The total of the covered numbers is $3x$. Once again the total of the **uncovered** numbers is $136 - 3x$, which is one more than a multiple of three no matter the value of $x$.

This means that no matter where Tile A is placed on to the grid, it is impossible for the total of the **uncovered** numbers to be a multiple of three.

(b) For the total of the **uncovered** numbers to be a multiple of three we need the total of the covered numbers to be one more than a multiple of three. This is because the total of all sixteen numbers in the grid is one more than a multiple of three.

We employ the same strategy as in part (a) but this time there are four orientations to consider. The diagrams below illustrate the four different cases.




The totals for the covered numbers in each orientation are are $3x - 5$, $3x + 3$, $3x + 5$ and $3x - 3$ respectively. Only $3x - 5$ is one more than a multiple of three, which can be seen by rewriting the expression as $3(x - 2) + 1$. This means that for the total of the **uncovered** numbers to be a multiple of three we must place Tile B in the first orientation.

Next we count in how many ways Tile B can be placed on to the grid in this orientation. If we consider which values of $x$ allow the tile to fit on to the grid we can see that only 6, 7, 8, 10, 11, 12, 14, 15 and 16 work. This means there are nine ways to place Tile B on the grid such that the total of the **uncovered** numbers is a multiple of three.

## NOTE

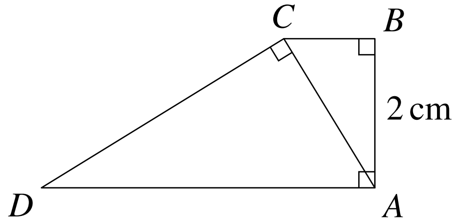
If you are familiar with modular arithmetic then you could rephrase and streamline each of the arguments above using the notion of congruence modulo 3.

This means that you replace the numbers in the grid by the remainders they give when divided by three. This yields the diagram on the right

| | | | |
|---|---|---|---|
| 1 | 2 | 0 | 1 |
| 2 | 0 | 1 | 2 |
| 0 | 1 | 2 | 0 |
| 1 | 2 | 0 | 1 |

You can now check the totals of covered numbers with Tile B in the four orientations. The first orientation gives a total of 1 or 4, both of which give remainder 1 when divided by 3. The second orientation gives a total of 3, the third a total of 2 or 5 (both of which give remainder 2) and the fourth gives a total of 3. Thus only in the first orientation is the total of covered numbers one more than a multiple of 3.

**3.** The diagram shows a quadrilateral $ABCD$, where $AB$ is 2 cm and $\angle ABC$, $\angle ACD$ and $\angle DAB$ are right angles.




(a) Let $E$ be the point on $DA$ such that $CE$ is perpendicular to $DA$. Prove that triangles $ABC$ and $DEC$ are similar. (2 marks)

(b) Given that the area of quadrilateral $ABCD$ is 6 cm$^2$, find all possible values for the perimeter of quadrilateral $ABCD$. (8 marks)

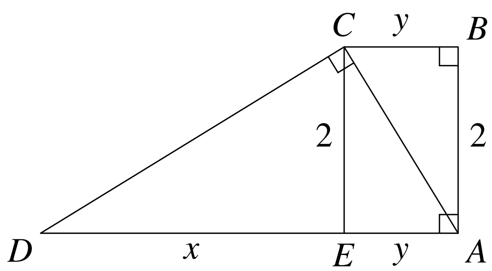
SOLUTION

COMMENTARY

In many geometry questions it is useful to start by thinking about what we are being asked to find, and then think about what related information is given in the question. In this case we want to find the perimeter of the shape, so we need to determine the lengths of the sides $BC$, $CD$ and $DA$. This looks like a lot of unknown lengths, so it would be useful if they were related to each other in some way. Note also that the question suggests that there is more than one answer.

Part (a) suggest drawing the height from $C$. Notice that this doesn't introduce any new unknown length, as $CE = 2$. Moreover, it creates a right angled triangle $DEC$, which allows us to express $CD$ in terms of the other lengths.

Now let's define some variables. There are various ways to do this, but we are going to define $DE = x$ and $EA = y$, so that $BC = y$ and $CD = \sqrt{x^2 + 2^2}$. This tells us that, in order to find the required perimeter, we only need to find $x$ and $y$.




At this point it is useful to read the question again and see what information we have about $x$ and $y$. Hopefully you have realised that $ABCD$ is a trapezium and you can write an equation for its area; this gives one equation connecting $x$ and $y$.

There are lots of right angled triangles in the diagram, so you could try writing some Pythagoras equations. It is possible to solve the problem in this way, as shown in the Alternative solution below, but it involves several quadratic equations. Part (a) suggests an alternative: using similar triangles. They are useful because they give you information about the ratio of sides, so you often get simpler equations than you get from Pythagoras.

To do part (a), you need to remember that two triangles are similar if they have equal angles. You don't know how big the angles actually are, but you can give one of them a name, such as $\theta$ and then express all the other angles in terms of $\theta$. Remember that you need to state which geometrical rules you are using in your calculation.

Once you have shown that the two triangles are similar, you should write down the ratios of the corresponding sides, giving you three equal ratios. You should then think about which one you can combine with your area equation; remember that you are looking for an equation involving $x$ and $y$.

(a) Add the point $E$ as in the diagram above, and let $\angle CDE = \theta$. Then we can do the following angle calculations:

$\angle DCE = 90° - \theta$ because angles in triangle $DCE$ add up to 180°.

$\angle ECA = \theta$ because $\angle DCE + \angle ECA = 90°$.

$\angle CAB = \angle ECA = \theta$ because they are alternate angles.

(It is also possible to use triangles $CEA$ and $ABC$ to work out remaining angles.)

Therefore triangles $ABC$ and $DEC$ are both right angled, and one of their other angles equals $\theta$. This means that the third angle is also equal, so they are similar.

(b) Define $x = DE$ and $y = EA$ as in the diagram above. Then $BC = y$ and $CE = 2$ because $ABCE$ is a rectangle.

From the similarity of triangles $ABC$ and $DEC$ we know that

$$\frac{AB}{DE} = \frac{BC}{EC}, \text{ i.e. } \frac{2}{x} = \frac{y}{2},$$

which is equivalent to $xy = 4$.

On the other hand, $ABCD$ is a trapezium (since the right angles at $A$ and $B$ imply that $AD$ and $BC$ are parallel), so its area is

$$\frac{(x + 2y)}{2} \times 2 = 6.$$

This gives another equation for $x$ and $y$: $x + 2y = 6$.

Substituting $x = 6 - 2y$ into $xy = 4$ and simplifying gives $y^2 - 3y + 2 = 0$ which has solutions $y = 1$ and $y = 2$. The corresponding $x$ values are $x = 4$ and $x = 2$.

The perimeter of $ABCD$ equals $x + 2y + 2 + \sqrt{x^2 + 4}$, so its two possible values are $8 + 2\sqrt{5}$ and $8 + 2\sqrt{2}$.

Alternative

It is possible to solve part (b) without reference to similar triangles, instead we can use several applications of Pythagoras' Theorem to derive the equation $xy = 4$. In addition to the variables defined above we define $u = CA$ and $v = CD$. Considering triangles $ABC$, $DEC$ and $DAC$ in turn and applying Pythagoras' Theorem to each yields the system of equations:

$$2^2 + y^2 = u^2 \tag{1}$$
$$x^2 + 2^2 = v^2 \tag{2}$$
$$u^2 + v^2 = (x + y)^2 \tag{3}$$

Adding equation (1) to equation (2), we obtain

$$8 + x^2 + y^2 = u^2 + v^2 \tag{4}$$

Subtracting equation (4) from equation (3) and rearranging and simplifying, we obtain

$$xy = 4$$

We may now continue as in the first solution.

**4.** Sam is playing a game. Her teacher gives her a positive whole number $A$, and then Sam chooses a positive whole number $S$. Sam then adds together all of the integers between $S$ and $S+A-1$ (inclusive) to obtain a total $T$. If $T$ is even, Sam wins the game. For example, if $A = 4$, Sam can win by choosing $S = 10$ because then $T = 10 + 11 + 12 + 13 = 46$.

(a) (i) Show that if $A = 4$, Sam will win the game no matter which number she chooses.

(ii) Show that if $A$ is a multiple of 4, Sam will win the game no matter which number she chooses. (3 marks)

(b) For which other values of $A$ can Sam choose an $S$ so that she wins? You must show how she can win for each of those values, and also explain why she cannot win for all the other values. (7 marks)

## Solution

### Commentary

As with all "games" questions, it is a good idea to start by playing the game a few times to make sure you understand the rules and get a feel for how it works. For example, you may realise that $A$ represents how many numbers Sam is adding and $S$ represents the starting number. You may also get some idea of when Sam can win.

It is also worth reading the whole question straight away. Parts (a) and (b) are worded differently, suggesting that for some values of $A$ the total will always be even (regardless of the starting number), while for other values of $A$ Sam will need to choose the starting number to try and make the total even.

There are two main ways to approach this problem. If you are confident with sums of series, you may be able to find a formula for $T$ in terms of $A$ and $S$ (this is just the sum of $A$ consecutive numbers, starting from $S$). You can then analyse this formula to see which combinations of $A$ and $S$ make $T$ even. This strategy is used in the Alternative solution below.

However, the knowledge of sums of series is not required. Part (a) is designed to suggest that adding numbers in groups of four may be useful. Can you write an algebraic expression for the sum of four consecutive numbers? What happens if you then have several groups of four? This suggests that for part (b) you need to consider which numbers are left over, and so the answer may depend on the remainder when $A$ is divided by 4.

The final sentence of part (b) reminds you that you need to discuss all the different cases when answering this type of question: If you think that Sam can win, you need to say which number she needs to choose; if you think that she can't win, you need to demonstrate that every value of $S$ gives an odd total.

(a) (i) When $A = 4$, the total is $T = S + (S + 1) + (S + 2) + (S + 3) = 4S + 6$, which is even for all values of $S$. Therefore Sam will win no matter which value of $S$ she chooses.

(ii) If $A$ is a multiple of 4, the numbers can be grouped into groups of four with no numbers remaining. We know from part (i) that four consecutive numbers always have an even total, and adding several even numbers always gives an even number. So $T$ will always be even in this case.

(b) If $A$ is not a multiple of 4, Sam can form groups of four consecutive numbers as before to make even totals, but now there will be one, two or three numbers remaining. The total of these remaining numbers will determine whether $T$ is even or odd. We can group the numbers so that the ungrouped numbers are at the start. (For example, numbers 1 to 10 can be grouped as $1 + 2 + (3 + 4 + 5 + 6) + (7 + 8 + 9 + 10)$.)

If $A$ is one more than a multiple of 4 then the only remaining number is $S$. Sam can win by choosing $S$ to be an even number.

If $A$ is two more than a multiple of 4 then the remaining numbers are $S$ and $S + 1$. These add up to $2S + 1$ which is always odd, so Sam can not win in this case.

If $A$ is three more than a multiple of 4 then the remaining numbers add up to $S + (S + 1) + (S + 2) = 3S + 3$. This is even when $S$ is odd, so in this case Sam can win by choosing an odd number.

Therefore Sam can win when $A$ is one or three more than a multiple of 4 (i.e. when $A$ is odd).

ALTERNATIVE

The sum of the $A$ numbers from $S$ to $S + A - 1$ is

$$T = \frac{A(2S + A - 1)}{2}.$$

(a) If $A = 4k$ (this includes the case $A = 4$) then $T = 2k(2S + 4k - 1)$, which is even for all $S$. Therefore Sam wins no matter which value of $S$ she chooses.

(b) If $A$ is odd, so $A = 2k + 1$, then $T = (2k + 1)(S + k)$. Sam can make the second bracket even by choosing $S$ odd when $k$ is odd and $S$ even when $k$ is even. Therefore Sam can win whenever $A$ is odd.

Finally, if $A$ is even but not a multiple of 4, so that $A = 4k+2$, then $T = (2k+1)(2S+4k+1)$. Both brackets are odd for all values of $k$ and $S$, so in this case Sam can not find a winning number.

**5.** (a) By considering their difference, or otherwise, find all possibilities for the common factors of $n$ and $n + 3$. (1 mark)

For $n \geq 2$, let $P(n)$ denote the largest prime factor of $n$.

(b) If $a$ and $b$ are positive integers greater than 1, explain why $P(ab)$ must be equal to at least one of $P(a)$ or $P(b)$. (1 mark)

(c) Find all positive integers $n$ such that $P(n^2 + 2n + 1) = P(n^2 + 9n + 14)$. (8 marks)

## SOLUTION

### COMMENTARY

This is a difficult problem, which requires you to put together lots of different ideas. As usual, it is useful to start by looking at some examples to understand what the function $P(n)$ does and what the question is asking you to do. Maybe you can try some values of $n$ and see if you can find any that satisfy the equation in part (c).

The introductory parts suggest two of the main ideas involved in solving this problem.

Part (a) asks you to think about common factors of two numbers. It should be clear that a common factor of two numbers cannot be larger than their difference: for example, if two numbers differ by 4, they can't both be multiples of 5. This observation is useful for limiting the options for common prime factors in part (c).

Part (b) reminds you about an important property of prime numbers: if a prime divides a product of two numbers, then it must divide at least one of the numbers. It is important to note that this is only true for prime divisors. For example, if 15 divides $ab$, it does not necessarily divide either $a$ or $b$: it could be the case that 3 divides $a$ and 5 divides $b$ (e.g. consider $a = 6$, $b = 10$, $ab = 60$).

Part (b) suggests that you should begin part (c) by factorising the two expressions in brackets. Even without this hint, the question is about finding values of $n$ such that the two expressions have the same largest prime factor, so factorising is definitely the way to start. The result of part (b) then allows you to look at different cases involving separate linear factors.

One of the trickiest steps in solving this problem is realising that, instead of focussing on what $n$ can be, you should think about the possible values of the largest common prime factor of two expressions. This is where the idea from part (a) comes in: you have two expressions that differ by a small number, so this limits what their common prime factors can be.

Even after you have found the possible prime factors of $n^2 + 2n + 1$ and $n^2 + 9n + 14$, there is still a final twist: you must ensure that these are the *largest* prime factors of both expressions. This requires some careful thinking, but at the very least you should check that any solutions you have found do actually work, by checking the values of $P(n^2 + 2n + 1)$ and $P(n^2 + 9n + 14)$.

(a) Suppose $d$ divides both $n$ and $n + 3$. Then $d$ must also divide their difference, which is 3. Therefore $d$ can only be 1 or 3.

(b) Let $p = P(ab)$. This means that $p$ divides $ab$. Since $p$ is prime, it must divide at least one of $a$ and $b$; suppose that $p$ divides $a$.

If $a$ had a prime factor larger than $p$, that factor would also divide $ab$, so $p$ would not be the largest prime factor. Therefore $p$ is the largest prime factor of $a$, which means that $p = P(a)$.

(c) We are looking for $n$ such that $P((n + 1)^2) = P((n + 2)(n + 7))$. From part (b),

$$P((n + 1)^2) = P(n + 1)$$

and

$$P((n + 2)(n + 7)) = P(n + 2) \text{ or } P(n + 7).$$

Now, by the argument used in part (a), $n + 1$ and $n + 2$ have no common factors greater than one (since their difference is one), so the only option is that

$$P(n + 1) = P(n + 7).$$

Since $n + 1$ and $n + 7$ differ by 6, all their common factors must divide 6. So the only possibilities are

$$P(n + 1) = P(n + 7) = 2 \text{ or } 3.$$

We must also ensure that this is the *largest* prime factor of both $n + 1$ and $(n + 2)(n + 7)$. In particular, all prime factors of $n + 2$ must be smaller than $P(n + 1)$.

This rules out the case $P(n + 1) = P(n + 7) = 2$, as then any prime factor of $n + 2$ would be larger than 2. The only remaining possibility is that $P(n + 1) = P(n + 7) = 3$ and the only prime factor of $n + 2$ is 2.

In this case, $n + 2$ is even so $n + 1$ and $n + 7$ are odd and their only prime factor is 3. In other words, they are both powers of 3.

The only powers of 3 that differ by 6 are 3 and 9 (since the difference between consecutive powers of 3 increases, we only need to check the first few cases), which gives that the only possible value of $n$ is 2. We can check that $n = 2$ indeed gives a solution, since if $n = 2$ then $(n + 1)^2 = 9$ and $(n + 2)(n + 7) = 36$, and $P(9) = P(36) = 3$.