UK Maths Trust

Mathematical Olympiad for Girls

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. They are not intended to be the 'best' possible solutions; in some cases we have suggested alternatives, but readers may come up with other equally good ideas.

All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or been unclear in your explanation, then you will normally receive nearly all the marks.

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1. This question requires full written explanations.

$ABCD$ is a quadrilateral, with vertices labelled in anti-clockwise order, such that:

AB is parallel to DC,

$AB = AC$, and

angle ADC is equal to angle ACB.

(a) Draw a diagram to show this information. Your diagram need not be to scale, but you should mark clearly equal lengths and angles.

(2 marks)

(b) (i) Prove that $AD = BC$ and that AD is parallel to BC.

(ii) What type of quadrilateral is $ABCD$? (8 marks)

COMMENTARY

(a) Even if the question had not asked you for a diagram, drawing a clear diagram should have been your first instinct. The question asks you to clearly indicate equal sides and angles and it is also a good idea to mark the pair of parallel sides. If you do draw a diagram that is roughly to scale you will probably get a clue about the type of shape that you get, but it is important that you use geometric reasoning rather than making an assumption based on what the diagram looks like.

(b) If you mark the equal sides carefully then you should spot that ACB is an isosceles triangle. Angle chasing, using angle sum of triangle and properties of parallel lines, should give you that $\angle CAD = \angle CDA$, which means that CAD is also isosceles.

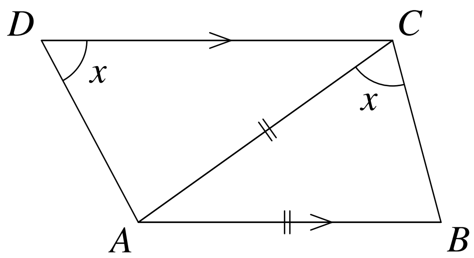
The pair of isosceles triangles have the same angles, and share an equal side so they are congruent. This allows one to conclude that $AD = BC$.

The most subtle part of the question is how you to establish that AD and BC are parallel. You probably know lots of relationships that must be true when you have a pair of parallel lines, for example you will know that pairs of alternate angles are equal. Here you will need to think about how to argue in the opposite direction: what do you need to show to be true in order to justifiably conclude that the two lines are parallel?

The argument below uses the *converse* result that if a pair of alternate angles are equal then the lines must be parallel. You should think about what other facts would have been sufficient to check.

Solution

(a)



Note that we clearly indicate the pair of equal lengths AC and AD, the pair of parallel sides AB and CD, and the equal angles $\angle ACD$ and $\angle ADC$. It makes sense to mark these equal angles as x.

(b) We firstly note that because $AB = AC$, the triangle ABC is isosceles, and so

$$\angle ABC = \angle ACB = x.$$

By considering the sum of the angles in the triangle ABC we have that $\angle BAC = 180° - 2x$. Since AB and DC are parallel, we have that $\angle DCA = \angle BAC$ (alternate angles).

Now as $\angle ADC = x$ and $\angle DCA = 180° - 2x$, by considering the angles in the triangle ACD we have $\angle CAD = 180° - x - (180° - 2x) = x$. Because $\angle CAD = \angle ADC = x$, the triangle ACD is isosceles. We hence have $DC = AC$.

The two isosceles triangles ACD and CAD share the same angles (x, x and $180° - 2x$) and $DC = AC = AB$. Hence triangles CAD and ABC are congruent. We can thus conclude that $AD = BC$.

Furthermore because the pair of alternate angles CAD and ACD are equal, we can conclude that AD and BC are parallel. Note the use of the *converse* theorem at this point.

(ii) As both pairs of opposite sides are parallel, $ABCD$ is a parallelogram.

Note

The fact that $ABCD$ is a parallelogram is perhaps the worst kept secret in the world, and you may well have thought this was "obvious" even from the point of having drawn the diagram. The key challenge in this question is to make sure that you successfully give an argument that *guarantees* that $ABCD$ is a parallelogram.

2. This question requires answers only.

In this question, $\overline{abc}$ denotes a three-digit number with digits a, b, c.

(a) Write down all three-digit multiples of 3 which only contain digits 1, 2 and 3. Digits can be repeated.

(3 marks)

(b) (i) Write down the values of b for which $9b^b < 1000$.

(ii) Let a, b and c be non-zero digits. Find all three-digit numbers $\overline{abc}$ which satisfy the equation

$$3c^c + 6a^a + 9b^b = \overline{abc}.$$

(7 marks)

COMMENTARY

(a) In the worst case there are only 27 numbers that need to be checked. But is there a fact about which numbers are divisible by three which comes to your aid?

(b) (i) If you calculate $1^1, 2^2, 3^3, \ldots$ you will see that the numbers become very large very quickly, and only a small number of values of b are possible for which $b^b < 1000$.

(Note that 0^0 is undefined, so we are not including $b = 0$ as an option.)

(ii) Noting $3c^c + 6a^a + 9b^b$ is a three digit number, part (i) means that we can rule out all but a small number of values of b. Modifying the argument slightly means that we can also show that a and c must also be small. Can you now see why the hint in part a) is useful?

Once you have reduced the number of possibilities to a manageable number, there is no way to avoid simply using brute force to check which ones work and which ones don't.

Although this is an answer-only question, we present a solution as would have been required in a full-answer question.

SOLUTION

(a) A very useful fact that can be quoted in Olympiad papers is that a number is divisible by 3 if and only if the digit sum is divisible by 3. For a three digit number made up of ones, twos and threes, the only possible digit sums are three, six and nine. A digit sum of 3 is only possible with three ones, a digit sum of 9 is only possible with three threes, whereas six can either be made by three twos, or any of the six orderings of a one, a two and a three.

The desired numbers are thus:

$$111, 222, 333, 123, 132, 213, 231, 312, 321$$

(b) (i) As $9 \times 3^3 = 243 < 1000$ and $9 \times 4^4 = 2304 > 1000$ the only possible values of b for which $9 \times b^b < 1000$ are $b = 1, b = 2$ or $b = 3$.

(ii) We note that because $3c^c + 6a^a + 9b^b$ is a three digit number, we must have $9b^b < 1000$. So from the previous part b can only be equal to one, two or three.

Because $3 \times 4^4 = 768$, $c \geq 4$ implies $\overline{abc} \geq 768$ meaning $a \geq 7$. However this is impossible because $6 \times a^a \geq 6 \times 7^7 > 1000$, which contradicts the fact that $3c^c + 6a^a + 9b^b$ is a three digit number. This means that $c \leq 3$. We can show $a \leq 3$ by a very similar argument.

We therefore have that a, b and c must all be equal to one, two or three. As each part of the the sum $3c^c + 6a^a + 9c^c$ is a multiple of 3 we must have that $\overline{abc}$ is a multiple of 3. This means that the only numbers that could possible work are the nine numbers from part (a). Checking each in turn we see that:

$$\begin{aligned}
3 \times 1^1 + 6 \times 1^1 + 9 \times 1^1 &= 18 \neq 111 \\
3 \times 2^2 + 6 \times 2^2 + 9 \times 2^2 &= 72 \neq 222 \\
3 \times 3^3 + 6 \times 3^3 + 9 \times 3^3 &= 486 \neq 333 \\
3 \times 3^3 + 6 \times 1^1 + 9 \times 2^2 &= 123 = 123 \\
3 \times 2^2 + 6 \times 1^1 + 9 \times 3^3 &= 261 \neq 132 \\
3 \times 3^3 + 6 \times 2^2 + 9 \times 1^1 &= 114 \neq 213 \\
3 \times 1^1 + 6 \times 2^2 + 9 \times 3^3 &= 270 \neq 231 \\
3 \times 2^2 + 6 \times 3^3 + 9 \times 1^1 &= 183 \neq 312 \\
3 \times 1^1 + 6 \times 3^3 + 9 \times 2^2 &= 201 \neq 321
\end{aligned}$$

We can see that 123 is the only number that works.

3. This question requires answers only.

(a) Five identical coins are placed in the cells of a 3×3 grid so that there is at most one coin in each cell and there is an odd number of coins in each row and each column.

(i) Show two examples of how this could be done.

(ii) In how many ways can this be done?

(3 marks)

(b) The numbers 1 to 9 are arranged in the cells of a 3×3 grid so that every row and every column have an odd sum.

(i) Show two examples of how this could be done.

(ii) How many such arrangements are possible?

You do not need to multiply out your answer, and may write it as a product, such as $2 \times 7 \times 289$ or 3×17.

(7 marks)

Commentary

If you are not sure how to start, it is always a good idea to try a few examples to make sure you understand the rules for each of the parts. In part (a), how can you distribute the five coins so that the number in each row and each column is odd?

A common mistake in questions of this type is to start by thinking: "If I put the first coin in the corner and put the second coin next to it, then the third coin has to go in the same row" etc. This will probably lead you to the systematic listing of all possibilities, which could work if the answer is fairly small, but quickly becomes unmanageable and prone to errors.

Instead, it is more efficient to think of some common properties that are shared by all allowed arrangements of coins, and use this to restrict the number of things you need to count. Another really useful tip for questions that involve counting is to imagine actually placing the coins: What would you do first?

For example, how can the five coins be distributed among the three rows? There is basically only one way: One row with three coins and two rows with one coin each. The only thing you can change is which of the rows is the one with three coins. Once you have fixed the row with three coins, the remaining two coins can be moved in their rows, but they need to remain in the same column (otherwise you would create two columns with two coins each, which is not allowed). So you need to place the remaining two coins in the same column. Can you now count how many ways there are to do that?

For part (b), it is tempting to start by thinking what possible odd sums you can get. But you will soon realise that there are two many options. So again, instead of thinking about the specific value of the sum, just focus on the fact that it needs to be odd: If the sum of the three numbers in a row is odd, what could those three numbers be?

This hopefully leads you to see the connection with part (a); odd numbers are like the coins – you can have either one or three of them in each row, and there are five of them in total. However, the final answer is not the same as the answer to part (a) because, while in part (a) all the coins were identical, the five odd numbers are all different. Furthermore, the remaining four cells, which in part (a) were empty, now contain the even numbers.

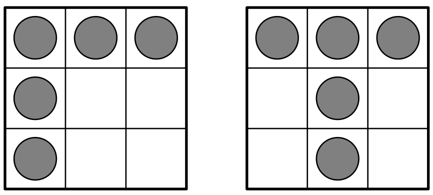
To complete the count, it helps again to think about producing one such arrangements: First you select the five cells for the odd numbers (this is the answer from part (a)); then you need to count in how many ways you can rearrange the five odd numbers among those five cells, and in how many ways you can rearrange the four even numbers among the remaining four cells.

You may already know how to count all possible arrangements of n objects, in which case you can just write down the answer. If you don't know this, you can again think about placing the odd numbers into the five selected cells. Number 1 can be placed in any of the five cells; then number 3 can be placed in any of the remaining four cells; number 5 in any of the remaining three cells; number 7 in any of the remaining two cells; and finally there is only one possible cell left for number 9. Using the "product rule for counting", the total number of possible arrangements of the five odd numbers is $5 \times 4 \times 3 \times 2 \times 1$. This number is denoted by 5! (pronounced "5 factorial"). You then need to remember to count the possible arrangements of the four even numbers in a similar way, and finally use the product rule for counting again to get the final answer. Note that the question says that you do not need to calculate the actual number, and the answer below is given as a product.

As above, although answers only are required to obtain full marks, we present a solution as would have been written for a long-answer question.

Solution

(a) Here are two possible examples:



Each row and each column can contain either one or three coins. Since there are five coins in total, two of the rows need to contain one coin each and the remaining row needs to contain three coins; the same is true for the columns.

Once we have placed three coins in one of the rows and three coins in one of the columns, all five coins have been used. So we just need to count the number of ways to choose the row and the column that contain three coins.

There are three choices for the row; for each of those choices, there are three choices for the column. So the total number of choices is $3 \times 3 = 9$.

(b) Here are two possible examples:

1	3	5
7	2	4
9	6	8

1	3	5
7	8	6
9	4	2

There are two ways to make an odd sum using three numbers: three odd, or one odd and two even. So each row and each column need to contain either one or three odd numbers. There are five odd numbers between 1 and 9 inclusive, so we can make a link to part (a): The number of ways to choose the cells that contain odd numbers is 9.

For every choice of odd-numbered cells, there are five odd numbers to arrange in those five cells, and four even numbers to arrange in the remaining four cells. The number of ways to do this is $5! \times 4!$.

Therefore the total number of possible arrangements of the numbers is $9 \times 5! \times 4!$.

4. This question requires full written explanations.

A *Pythagorean triple* is a triple (x, y, z) of positive integers satisfying $x^2 + y^2 = z^2$. A triple is called *primitive* if no factor greater than 1 is shared by all of x, y and z.

(a) Show that, for every positive integer $n \geq 4$, $x = n^2 - 9$, $y = 6n$ and $z = n^2 + 9$ form a Pythagorean triple. Find one value of n for which this triple is *not* primitive.

(2 marks)

(b) Show that, in any Pythagorean triple, x and y cannot both be odd.

(2 marks)

(c) Find all primitive Pythagorean triples in which two of x, y and z differ by two.

Your solution must show that all the triples you found are primitive and that there are no other possibilities.

(6 marks)

You may not quote any general formulae for Pythagorean triples without proof.

Commentary

This question may look intimidating at the first sight, as it starts by introducing some unfamiliar terminology. It also asks you to find all possible solutions to a quadratic equation in three variables, which seems daunting. However, if you take some time to read it carefully, you will hopefully see that it also gives you a lot of hints about what is expected.

Part (a) should help you both test your understanding of the terminology (What equation does a Pythagorean triple satisfy? How can you tell whether a triple is primitive?) and suggest that the final answer is not necessarily a list of numbers, but can be described in an algebraic form (this is usually called a "general solution"). To find an example of a triple that is not primitive, you can simply try some values of n; it may help you to note that y is always a multiple of 2 and 3.

Part (b) requires you to do some algebra. Remember that an odd number can be written in the form $2k + 1$. You then need to decide what it is about the expression you get for $x^2 + y^2$ that makes it impossible for it to be a square number. Think about what factors it can or cannot have.

Before getting into the details of part (c), it is useful to think about what you will actually need to do. The requirement that two of the numbers differ by two means that you can reduce the number of variables by writing, for example, $y = x + 2$; this should lead to a simpler equation, but be careful to consider all possible options. (It may well be that there are several cases that may need to be analysed separately.) Part (a) suggests that the answer may well involve expressing all three variables in terms of some general integer n, so look for ways to do that. From part (b) you know that x and y cannot both be odd, so it seems a good idea to check whether they can both be even.

Putting all the above ideas together and doing some algebra should lead you to a general expression for x, y and z. It is now time to step back and assess what you have proved, and what still remains to be proved.

All the analysis above shows that *if* solutions exist, *then* they must be of the form you have found. This does not guarantee that every triple of that form is in fact a solution, but that can be checked by direct calculation. You also need to check which of your triples are primitive. To do this, you need to consider what common factors each pair of numbers could possibly have. The fact that two of the numbers differ by two restricts what common factors they can have, so it turns out there are not very many options to check.

Solution

(a) Clearly x, y and z are all positive integers when $n \geq 4$. We can check that

$$\begin{aligned} x^2 + y^2 &= (n^2 - 9)^2 + (6n)^2 \\ &= n^4 - 18n^2 + 81 + 36n^2 \\ &= n^4 + 18n^2 + 81 \\ &= (n^2 + 9)^2 = z^2. \end{aligned}$$

Hence x, y and z form a Pythagorean triple.

When $n = 6$, we have $(x, y, z) = (27, 36, 45)$. These are all multiples of 3 so this triple is not primitive.

(b) Suppose that x and y are both odd, and write $x = 2a + 1$ and $y = 2b + 1$. Then:

$$\begin{aligned} x^2 + y^2 &= (4a^2 + 4a + 1) + (4b^2 + 4b + 1) \\ &= 4(a^2 + a + b^2 + b) + 2. \end{aligned}$$

Hence $x^2 + y^2$ is a multiple of 2 but not a multiple of 4, so it cannot be a square number. It is therefore not possible that $x^2 + y^2 = z^2$ when x and y are both odd integers.

(c) There are two cases: either x and y differ by 2 or x and z differ by 2. (The case where y and z differ by 2 is analogous to the second case.)

If x and y differ by 2, from part (b) we know that they cannot both be odd, so both have to be even. But then z is also even and the triple is not primitive. Hence this case is impossible.

If x and z differ by 2, we can write $z = x + 2$ (as clearly $z > x$), and the equation becomes

$$\begin{aligned} x^2 + y^2 &= (x + 2)^2 \\ y^2 &= 4x + 4. \end{aligned}$$

Since y^2 is even, y is also even, so we can write $y = 2b$ for some integer b. Dividing the equation by 4 gives $b^2 = x + 1$ and hence we can write

$$x = b^2 - 1, \quad z = x + 2 = b^2 + 1.$$

We can check that this works for every integer $b \geq 2$ (as x needs to be positive):

$$\begin{aligned} x^2 + y^2 &= (b^2 - 1)^2 + (2b)^2 \\ &= b^4 - 2b^2 + 1 + 4b^2 \\ &= b^4 + 2b^2 + 1 \\ &= (b^2 + 1)^2 = z^2, \end{aligned}$$

as required. Hence, $(b^2 - 1, 2b, b^2 + 1)$ is a Pythagorean triple for all $b \geq 2$.

We need to check whether this triple is primitive. We can see that y is always even. When b is odd, x and z are also even, so the triple isn't primitive. When b is even, we can write $b = 2k$, with $k \geq 1$, so that the triple becomes

$$(x, y, z) = (4k^2 - 1, 4k, 4k^2 + 1).$$

Since x and z differ by 2, their only possible common factor could be 2. But they are both odd, and hence have no common factors.

In conclusion, $(x, y, z) = (4k^2 - 1, 4k, 4k^2 + 1)$ is a primitive Pythagorean triple for every positive integer k.

Note

It can be shown that *all* primitive Pythagorean triples are given by $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$ for some positive integers m and n. Can you work out which values of m and n give primitive triples? Can you see how to use this formula to get a triple such as $(9, 12, 15)$?

5. This question requires full written explanations.

(a) x, y and z are real numbers, with $x \leq y \leq z$ and

$$x + y = 4$$
$$y + z = 7$$

Let $T = x + y + z$.

(i) Show that $x \leq 2$ and that $T = 11 - y$.

(ii) Find the minimum possible value of T, giving one example of values of x, y and z where this occurs.

(2 marks)

(b) a, b, c, d and e are real numbers, with $a \leq b \leq c \leq d \leq e$ and

$$a + b + c = 4$$
$$b + c + d = 5$$
$$c + d + e = 9$$

Let $S = a + b + c + d + e$.

Find the minimum possible value of S, giving one example of values of a, b, c, d and e where this occurs.

Your solution must fully justify why no smaller value of S is possible.

(8 marks)

COMMENTARY

Perhaps more than any other question the neat, crisp solution presented below is likely to be extremely different to the problem solving process. It is not at all clear from the outset how the order relations $x \leq y \leq z$ and $a \leq b \leq c \leq d \leq e$ need to be used, and especially, in the second part, which has three equations in five unknowns, it is difficult to see which of the myriad of algebraic relationships between the variables are most useful in solving the problem.

You are likely to be familiar with pairs of simultaneous equations with two variables. However, in part a) you will notice that there are three variables, x, y and z. This presents an extra challenge. In fact, rather than get single values for each variable, the best that you can do is to get two of the variables (for example y and z) in terms of the third variable (for example x). You can then use the ordering $x \leq y \leq z$ to find upper and lower bounds for each of the variables. Depending on the approach used one can show one of the following facts.

$$\frac{1}{2} \le x \le 2$$
$$2 \le y \le \frac{7}{2}$$
$$\frac{7}{2} \le z \le 5$$

Part (a) then invites us to find the minimum of the total $T = x + y + z$. To show the minimum value of T is m you need to do two things, firstly that for every allowable value of x, y and z you have that $T \ge m$ (we say that m is a lower bound) and secondly that T can actually equal m for some choice of x, y and z. Failure to verify that the lower bound can be achieved would be a mistake. For example one may reason that as $x \ge \frac{1}{2}$, $y \ge 2$ and $z \ge \frac{7}{2}$ then adding shows that $x + y + z \ge 6$. This inequality is valid, but it turns out that it is not possible to minimise each variable simultaneously and a total of 6 is not possible.

The hint, however, gives us a big clue about how to simplify the problem, by inviting us to find an expression for T in terms of just one of the variables, in this case y. The expression $T = 11 - y$ shows that to minimise T we just need to maximise y. Using $y \le \frac{7}{2}$ and the corresponding values of x and y gives the minimum possible value of T that can actually be achieved.

Part (b) looks formidable and it is unlikely that many candidates have encountered simultaneous equations in five unknowns before. This however, is typical of Olympiad problem solving, as the questions will require you to deal with unfamiliar problems.

The hint from part a) suggests we should try and get an expression for S in terms of just one of the variables and adding the first and third equations allows one to conclude that $S = 13 - c$. This shows that we need to try and maximise c.

The problem with having so many variables is that there are a potentially bewildering range of things that you could do, and you will need to a cool head to come up with a strategy that leads to the answer. Perhaps one of the easiest observations, which follows by comparing the first and second equation, is that $d = a + 1$ and the relationship $c \le d$ gives that

$$c \le a + 1.$$

On its own this is not sufficient to find the maximum of c, and you will need to find a second constraint on c. As is common in Olympiad problems there is not one single approach, and as well as the method outlined below one could use $b \ge a$ together with $c = 4 - a - b$ to establish that

$$c \le 4 - 2a.$$

Once two inequalities have been found for c you can combine them to find an upper bound. If you can show that this can be achieved then you will have successfully maximised c, and this solution will also yield a minimum for T. The solution below gives a purely algebraic approach, but you could also think about graphical methods. For example how does the graph of the lines $y = 4 - 2x$ and $y = x + 1$ together with with the fact that $c \le 4 - 2a$ and $c \le a + 1$ help establish an upper bound for c?

Whatever method you use, you will see that c ends up being maximised in the special case where $c = d$ and $a = b$. This should not be a surprise, because, as $c \le d$, we can see that for a given value of d setting $c = d$ will make c as large possible. Similarly as $c = 4 - a - b$, then for a given value of a, making b as small as possible (by setting it equal to a) will also make c large. While this is a useful observation which puts our solution into some sort of context, it would not have been possible to solve this problem by *assuming* that $b = a$ and that $c = d$, and it was necessary to argue algebraically.

SOLUTION

(a) (i) Rearranging the first equation gives $y = 4 - x$, and using that expression in the second equation gives $(4 - x) + z = 7$ and hence $z = x + 3$

The relationship $x \le y \le z$ now becomes $x \le 4 - x \le 3 + x$, which yields:

$$\frac{1}{2} \le x \le 2.$$

(ii) Adding the two given equations we get $(x + y) + (y + z) = 4 + 7 = 11$, which simplifies to $T = 11 - y$. Thus to minimise T we want y to be as large as possible (which in turn means making x small). Using $x \ge \frac{1}{2}$ and $y = 4 - x$ allows us to conclude that $y \le \frac{7}{2}$. Using this in $T = 11 - y$ gives that $T \ge \frac{15}{2}$. We can check that this lower bound is achieved by the solution $x = \frac{1}{2}, y = \frac{7}{2}, z = \frac{7}{2}$.

(b) Using a similar idea to part (a), we can add the first and third equations to obtain $(a + b + c) + (c + d + e) = 4 + 9$, which simplifies to

$$S = 13 - c.$$

Thus to minimise S we thus want c to be as large as possible.

Subtracting the first equation from the second equation gives $(b+c+d)-(a+b+c) = 5-4$ which simplifies to $d = a + 1$. The relationship $c \le d$ gives

$$c \le a + 1.$$

Since $a \le b$ and $c \le a + 1$ we have that

$$c \le b + 1.$$

Adding the inequalities $c \le a + 1$ and $c \le b + 1$ gives $2c \le a + b + 2$, however from the first equation we can rewrite $a + b$ as $a + b = 4 - c$, leading to $2c \le 6 - c$, which gives

$$c \le 2.$$

Using $c \le 2$ and $S = 13 - c$ gives that $S \ge 11$.

Again we just need to check that this lower bound can be realised. Setting $a = 1, b = 1, c = 2, d = 2, e = 5$ satisfies both the three equations and the relationship $a \le b \le c \le d \le e$, and is such that $S = 11$.

ALTERNATIVE

It is also possible to use a graphical method to solve this problem.

We start as above by finding that $S = 13 - c$ and hence we need to find the maximum possible value of c. Using the three equations, we can write all the other variables in terms of a and c:

$$b = 4 - a - c, \quad d = a + 1, \quad e = 8 - a - c.$$

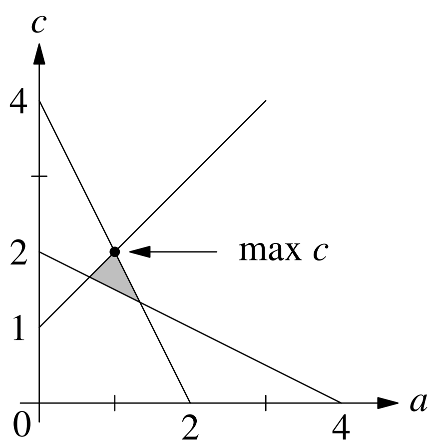
Then the inequalities become

$$a \le 4 - a - c \le c \le a + 1 \le 8 - a - c.$$

We can see that the final inequality follows from the first, so there are three inequalities defining the possible combinations of a and c:

$$2a + c \le 4, \quad a + 2c \ge 4, \quad c \le a + 1.$$

We can show all possible values of a and c on a graph by shading the region where all three are satisfied:



It is then clear that the maximum value of c occurs at the point where the lines $c = a + 1$ and $2a + c = 4$ intersect, which is at $a = 1, c = 2$.

Therefore the minimum possible value of S is $13 - 2 = 11$, and it occurs when $a = b = 1, c = d = 2, e = 5$.

NOTE

The graphical method has several variations, where we express S in terms of two of the variables, such as a and b. Finding the minimum values of S is then slightly trickier and requires a method called "linear programming".

