United Kingdom Mathematics Trust

email challenges@ukmt.org.uk
web www.ukmt.org.uk

Mathematical Olympiad for Girls 2024

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Introduction

For many students, and possibly some teachers, this is the first experience of attempting a Maths Olympiad paper. It may therefore be useful to understand how these papers are marked, as students may be disappointed to receive a small number of marks for a problem they thought they had almost solved.

Most questions in Olympiad papers are marked using what we call the '0+/10−' principle. This means that the markers first read the whole write-up and decide whether the student has a viable strategy to solve the problem. It may be that there are some mistakes or small gaps in their reasoning, but if those could be relatively easily filled in then this response is marked in the '10−' regime, with usually up to three marks being taken away for gaps and mistakes. Common examples of small gaps are algebraic or arithmetical errors (provided they don't change the nature of the argument), missing one of several cases in a counting question, or lack of geometrical reasons when calculating angles.

If, on the other hand, the student has only started to explore the problem and has only made some useful observations, but does not have a strategy to generalise or prove them, then the script is marked in the '0+' regime. Up to three marks may be available for spotting a pattern or trying an idea which, if progressed further, could lead to a solution. Examples of this in the present paper would be: in Question 2 splitting the count into cases (for example, by fixing the colour of circle A) and having a valid strategy for counting each case; in Question 3, using all the information to form an equation for a; in Question 5, proving that the difference between opposite numbers must be a multiple of 111. Notice that these all involve a substantial engagement with the problem, rather than just looking at some examples or special cases. Teachers should therefore reiterate to students that scoring even one or two marks on any of these questions is a real achievement.

The Mathematical Olympiad for Girls slightly different from other Maths Olympiads in that questions are broken down into several parts. Most of the time, the final part is the "main question" and the first part (or parts) are intended to suggest some useful results or good approaches to the problem. The reason for structuring the paper in this way is that the setters know that many of the candidates are not experienced in olympiad mathematics, and the hope is that by giving these pointers, we enable them to engage with the questions even if they are not familiar with standard olympiad techniques or "tricks". A useful hint is to read the whole question first and try to understand how the early parts may be helpful in solving the main problem.

The present paper also contains some answer-only questions, which nevertheless require a similar way of thinking to the usual olympiad questions. These questions are designed so that partial marks can be awarded to incorrect answers which show good progress towards the correct solution.

General comments

MOG continues to be a popular paper, and this year we yet again saw a large increase in the number of entries. Almost all candidates felt confident to tackle parts of each question, and the markers were yet again impressed by a large number of high-quality write-ups.

The questions this year were again based on common topics from the standard school syllabus, such as basic angle facts, similar triangles, algebraic factorisation and the product rule for counting. It was very pleasing to see so many students adapting their school knowledge to solve these more demanding problems.

While the knowledge required was relatively basic, careful thinking was needed in order to form a valid argument in all three full-solution questions. This was particularly true in Question 2, where most candidates knew that they need to multiply options for the colours of different circles, but still needed to be careful to consider different cases (for example, when D is the same or different colour from A). Question 3 was done particularly well, and it was really pleasing to see so many careful and detailed geometric arguments. Questions 1 and 4 obviously involved no written explanations, but it was still evident from the answers whether a candidate managed to make links between different parts of the question. Question 5 challenged even the strongest candidates, but it was good to see that many others managed to make some progress.

Overall, the standard of work seen was high and many of the full-response solutions were very well written.

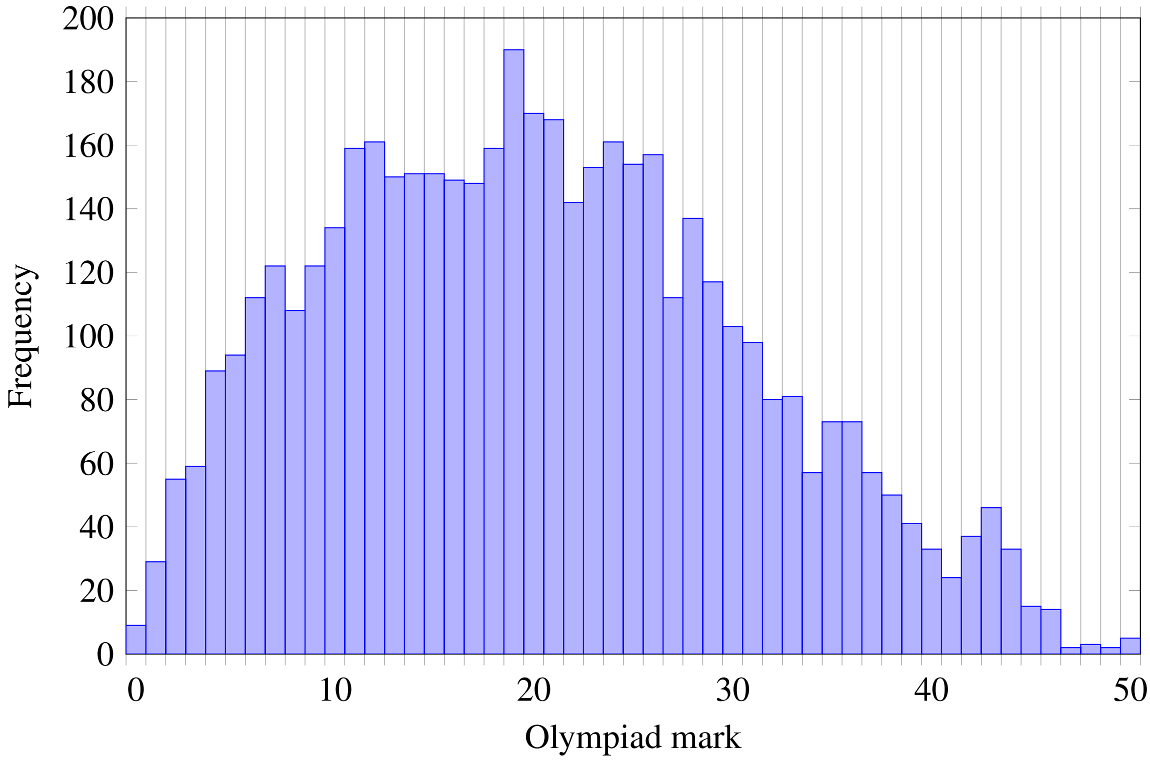
It is unfortunately often the case that students think that they have solved a problem but only receive two or three marks. There were two main examples in this year's paper. In question 2, the candidates who listed all thirty possible colourings (or even the ten possible colouring with the fixed colour of circle A) needed to explain carefully how they could be sure that they have not left out any. Many did this well by organising the options in a tree diagram, or carefully stated that they sorted them by the colour of B, then the colour of C, and so on. Those who listed the colourings in a non-systematic order were not able to score full marks, even when their list was fully correct.

The second example was Question 5, where most candidates did not realise the importance of the request to explain how the numbers, once paired up, could be arranged around the circle. Moreover, a large number of candidates showed that the pairings are possible when the difference is a factor of 111, but did not explain why no other differences worked. This type of omission is commonly seen in questions which ask to find all numbers with a certain property: you must show why your numbers work, as well as why no other number do. Of course, it may be that some candidates considered these questions and simply did not know how to answer them, and they should keep this in mind when tackling similar questions in future.

Mark distribution

The MOG 2024 paper was marked by a team of Abhilasha Aggarwal, Naomi Bazlov, Robin Bhattacharyya, Tom Bowler, Lin Chen, Andrea Chlebikova, John Cullen, Juliette Culver, Laura Daniels, Gareth Davies, Ana Meta Dolinar, Chris Eagle, Ben Fairfax, Chris Garton, Rebehah Glaze, Edward Godfrey, Anthony Goncharov, Aleksander Goodier, Amit Goyal, James Handy, Jon Hart, Alexander Hurst, Michael Illing, Vesna Kadelburg, Thomas Kavanagh, Emily Kenzie, Kit Kilgour, Jeremy King, Patricia King, Hayden Lam, Larry Lau, Aleks Lishkov, Thomas Lowe, Owen MacKenzie, Oliver Murray, Clavin Newn, Daniela Petti, Jay Rasdan, Heerpal Sahota, Amit Shah, Gurjot Singh, Alan Slomson, Geoff Smith, Marcus Smith, Rob Summerson, Stephen Tate, Cecilia Tudo, Joanna Tumelty, Tommy Walker Mackay, Kasia Warburton, Emma Wheeler, Dominic Yeo and Li Zhang.

We received responses from 4749 candidates, an almost 22% increase on last year.



The award thresholds were: 16 for Merit, 28 for Distinction and 46 for a Book Prize.

Question 1

For this question, follow the instructions on the answer sheet.

ABC is a right-angled triangle, with right-angle at B and side lengths AB and BC a whole number of centimetres.

F lies on AB, D lies on BC and E lies on AC, as shown in Figure 1 below, so that $BDEF$ is a rectangle.

Denote lengths, in centimetres, $BF = a$, $AF = x$, $BD = c$ and $CD = y$.

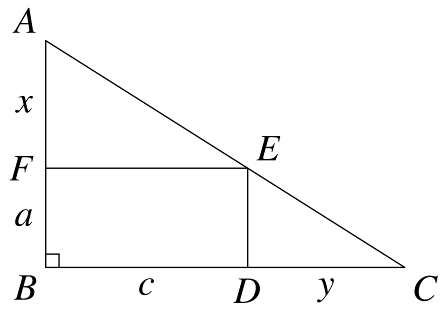


Figure 1

(a) Given that $a = 2$, $c = 6$ and $y = 3$, find x. (2 marks)

(b) It is now given that the area of rectangle $BDEF$ is 9 cm^2 and its dimensions are a whole number of centimetres.

Find all possible values for the length BC in centimetres. (8 marks)

Solution

(See the official solutions document.)

Markers' comments

This question was found demanding, with only around one-sixth of the candidates managing to fully solve it.

As with any answer-only question, it is difficult to tell exactly where the stumbling block was for unsuccessful candidates. However, judging by the presence of square roots in some of the answers, it is likely that some tried using Pythagoras instead of similar triangles to calculate the lengths; this is doable, but considerably more demanding.

Most candidates scored some marks for finding the three possible combinations of sides of rectangle $BDEF$, although some rejected the possibility of $a = c = 3$, forgetting that a square is a type of rectangle.

Question 2

This question requires full written explanations.

In Figure 2 below, each of the five circles A, B, C, D, E is to be coloured so that adjacent circles are different colours.

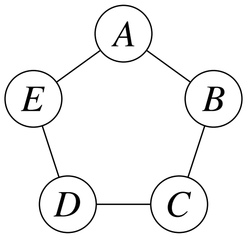


Figure 2

(a) In this part, circle A is to be coloured red and circle B yellow.

(i) Copy the diagram and show one possible way to colour the circles using colours red, yellow and green. Write 'R', 'Y' or 'G' in each circle to indicate its colour.

(ii) Explain why it is not possible to colour all five circles using only red and yellow. You should use labels A, B, C, D, E to refer to the circles.

(2 marks)

(b) In this part, the colours to be used are purple, orange and white. How many different ways are there to colour the circles using these three colours, so that adjacent circles are different colours? (8 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

Over 90% of candidates made some progress on this question, with over one-quarter solving it fully (with possibly some lost marks for lack of detail). On the other hand, almost half the candidates made some sort of fundamental error with their counting strategy. The most common errors are described below, and we hope that students can learn by thinking about them carefully.

It was clear that some candidates simply did not read the question properly, with their example not starting with A red and B yellow. Luckily this was a very small number, as was the number of those who did not understand the word 'adjacent' and tried answering a different question. However, a significant number forgot that in part (a)(ii), A was still red and B was yellow, and gave examples in which A was yellow.

To score the mark for part (a)(ii) it was necessary to explain why, starting with A red and B yellow, all the ways of colouring the circles lead to two adjacent circles having the same colour. With the question specifying the colour of the first two circles, this should have been an easy explanations if the candidates actually read the question carefully! It was not sufficient to just show an example of an incorrect colouring, without saying in what order the circles were coloured. In particular, it is not true that (in order $ABCDE$) RYRYR and YRYRY are the only possible options of colouring the circles, as one could start from, for example, B, and end up with RRYRYR. Many candidates tried to argue that the task is impossible because there is an odd number of circles, without saying what it is about an odd number that makes the task impossible.

It was really pleasing to see a wide variety of successful approaches to part (b) and many excellent and detailed explanations.

Many candidates went for the straightforward option of listing the ten possible colourings with A purple and then multiplying this by 3. Some used a tree diagram, which ensures that no cases are left out; however, it still needed to be said why six of the sixteen possible combinations were not counted (because they had E purple). Some simply tried to list the ten colourings; this made it harder to explain how it was ensured that all cases had been found, and a varying number of marks could be lost at different levels of detail. The same applies to solutions that listed all 30 possibilities; they could only score near to full marks if the listing was clearly systematic, for example sorted by the colour of A, then the colour of B, and so on.

Another popular approach is to try and use rotations, sometimes combined with the observation that two of the colours must repeat twice and the remaining colour once (this needed to be justified). For example, if there is a single purple in a fixed circle, there are only two ways to arrange the two oranges and two whites; we can then rotate this configuration five times, and also multiply by three for changing the single colour, giving $2 \times 5 \times 3 = 30$ possible colourings. A variation of this is trying to list six colourings that can be rotated to produce the rest. However, in this approach, it is crucial to check that none of the six are rotations of each other and, more importantly, explain how you know that their rotations produce everything else. Many candidates who went down this path clearly had a good intuition but were not able to justify it rigorously and did not score full marks.

It is possible to consider the options for the colour of each circle in turn, but we need to be careful how we combine different cases. Circle A can be any of the three colours, and circles B, C and D have two colour choices each. However, the number of options for E depends on whether D is the same colour as A or not. So we need to consider two separate cases, depending on the colour of D. If D is the same colour as A, then E has two options. However, for D to be the same colour as A, C must have been the colour different from both A and B, thus having only one choice rather than two. So the number of options for this case (with the numbers giving the options for each circle in order) is $3 \times 2 \times 1 \times 1 \times 2 = 12$. If D is a different colour from A, E has only one choice, D has two, and B and C together have three (for example, if A is purple and D is orange, the first four circles can be POWO, POPO and PWPO); so this case has $3 \times 3 \times 1 \times 2 = 18$ options. Hence the total number of colourings is $12 + 18 = 30$.

A large number of candidate who tried this approach ended up with the wrong answer. Some forgot that there are two different cases, thus getting either $3 \times 2 \times 2 \times 2 \times 2 = 48$ or $3 \times 2 \times 2 \times 2 \times 1 = 24$. Those who did identify the two options were usually not able to combine them correctly, either simply adding them together (to get the answer 72) or saying that they are equally likely and calculating $0.5 \times 48 + 0.5 \times 24 = 36$. In fact, as can be seen from the discussion above, the colourings with A and D the same colour account for two-fifths of all the colourings (12 out of 30).

Question 3

This question requires full written explanations.

(a) Figure 3 shows a semi-circle with centre O and diameter PQ and a point R on the semi-circle. By expressing other angles in the diagram in terms of p and q, prove that the angle in a semi-circle is 90°. You must clearly state any geometrical facts you use.

(2 marks)

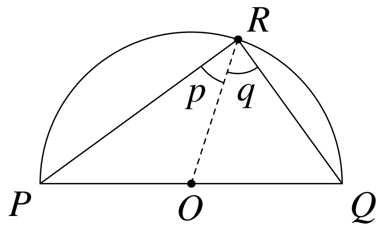


Figure 3

(b) Figure 4 shows an isosceles triangle ABC, with $AB = AC$. A semi-circle with centre M and diameter AC intersects AB at N.

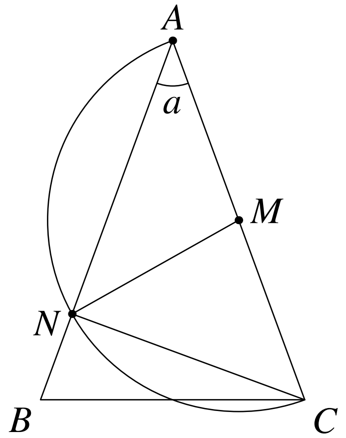


Figure 4

Given that $\angle AMN = 6\angle BCN$, find the size of angle $\angle BAC$, labelled a.

You must clearly state any geometrical facts you use. (8 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

This question was generally well answered, with almost half the candidates able to completely solve it, possibly with some minor lack of detail. It was pleasing to see so many engaging positively with geometry, which is traditionally the least popular olympiad topic.

In part (a), it was common for students to simply assert the statement or try to prove it by using more powerful results rather than follow the instructions to express other angles in terms of p and q. It was not entirely uncommon for candidates who got full marks in part (b) to get 0 in (a).

In part (b), it was impressive to see the variety of ways students connected the angles of different triangles and the multitude of equations reached. Some invoked the result from (a) and others did not. That said, many students did not make it clear which triangles they were considering in their working and this often meant students lost track of their thoughts. Similar problems arose when students introduced variables without explaining to themselves (and the reader) what they were. The best scripts led the reader through the different questions they were considering and justified each equation used by providing geometric reasons.

Usually, if candidates were able to combine enough facts about angles in the diagram, they proceeded to solve the problem completely, and almost all candidates who obtained a low mark here failed to spot or use an important geometrical idea. It was pleasingly rare to see a lack of justifications in combination with the correct answer.

A relatively common mistake was to misread which angle was six times another angle.

Question 4

For this question, follow the instructions on the answer sheet.

(a) The equation $uv + 3u - 3v - 18 = 0$ is equivalent to $(u - a)(v + b) = 9$. Write down the values of a and b. (1 mark)

(b) Hence find all integer solutions of the equation $uv + 3u - 3v - 18 = 0$. (3 marks)

(Note: 'integers' means whole numbers, which can be positive, negative or zero.)

(c) Find all integer solutions of the equation $x^2 - y^2 + 6y - 18 = 0$. (6 marks)

Solution

(See the official solutions document.)

Markers' comments

Over 90% of the candidates scored some marks on this question, and around three-quarters made some progress with parts (b) or (c). On the other hand, only around 10% solved it fully. Most scored some marks on both parts (b) and (c), possibly by finding a few of the solutions without a systematic method.

In part (b), it was very common to miss the negative factors of 9 and obtain only three solution pairs instead of six. A similar mistake was repeated in part (c) for candidates who chose the factorisation method, writing the equation as $(x + y - 3)(x - y + 3) = 9$. Those who opted for considering the difference of two squares, $x^2 - (y - 3)^2 = 9$, generally remembered negative numbers, but sometimes forgot that two squares that differ by 9 could be 9 and 0 as well as 25 and 16.

Candidates often made algebraic errors which led to incorrect solutions. These could have been detected simply by substituting them back – this is always good practice when solving equations. Olympiad papers are deliberately not too time pressured, so there should be plenty of time to check your work.

Question 5

This question requires full written explanations.

The positive integers from 1 to N are equally spaced around a circle in such a way that the sum of any two neighbouring numbers is equal to the sum of the two numbers diametrically opposite them.

(a) Figure 5 shows an example with $N = 6$. Number 1 is diametrically opposite number 4 and number 2 is diametrically opposite a.

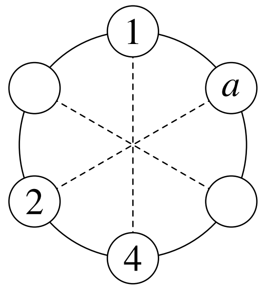


Figure 5

Copy and complete the diagram to show one possible way of arranging numbers 1 to 6. No explanation is needed in this part. (1 mark)

In the rest of this question, $N = 222$.

(b) Show that the difference between any pair of diametrically opposite numbers is the same. You may find it helpful to refer to Figure 6 in your explanation. (1 mark)

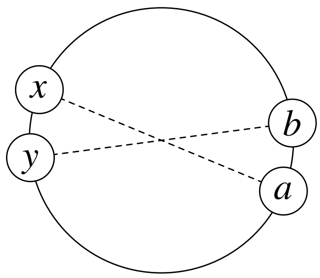


Figure 6

(c) Suppose that there is an arrangement in which number 41 is opposite number 1. By considering which numbers would need to be placed opposite 40, 120 and 200, show that such an arrangement is not possible. (1 mark)

(d) Find all possible numbers which could be placed opposite number 1. You need to explain why those are the only possibilities. In each case, you also need to explain why it is possible to arrange the remaining numbers. (7 marks)

Solution

(See the official solutions document.)

Markers' comments

It was heartening to see a large majority of candidates attempting question 5, with most of them scoring marks in the first three parts of the question! Many candidates who only attempted part (a) could have probably been a bit braver and tried the next two parts

There were some unfortunate misconceptions that came up a few times: for example, some candidates interpreted the "pair of diametrically opposite numbers" to mean a pair of adjacent numbers, and the corresponding diametrically opposed pair, rather than a single number on each side of the circle.

We were very happy with the overall standard of explanation in part (c), although some students hadn't realised $N \neq 6$ and were very confused. Candidates who did not use the difference idea from (b) and attempted to rely solely on the original criterion about sums were sadly not able to make correct progress. Another common mistake in was to say a difference of 40 meant 40 had to be paired with $40 - 40 = 0$, so obviously this is impossible.

Part (d) was a very demanding question, with only 20 making significant, although around 10% made some useful observations. Many students hypothesised that 112 was a correct possibility, concluding based on the diagram shown in part (a) that only a difference of $\frac{N}{2}$ is allowed. Spotting the arrangement for $N = 6$ with difference 1 between opposite numbers (which was drawn in some scripts) would have helped avoid this logical fallacy.

Quite a lot of students didn't quite get the distinction between showing all factors of 111 would work, and explaining why other differences wouldn't work. Quite a few wrote down it had to be a factor of 111 "by something similar to part (c)" and that wasn't enough of an explanation.

A very common problem was that, after finding (and possibly correctly proving) which numbers could go opposite 1, candidates did not attempt to justify the feasibility of these cases by describing how to arrange the 222 numbers around the circle – even though the question explicitly mentioned that this should be done. This final step is important because the task is actually impossible when N is a multiple of 4, as can be seen by, for example, by trying to produce a valid arrangement with $N = 8$.