UK Maths Trust

Mathematical Olympiad for Girls

supported by Jane Street Overleaf

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. They are not intended to be the 'best' possible solutions; in some cases we have suggested alternatives, but readers may come up with other equally good ideas.

All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or been unclear in your explanation, then you will normally receive nearly all the marks.

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1. For this question, follow the instructions on the answer sheet.

ABC is a right-angled triangle, with right-angle at B and side lengths AB and BC a whole number of centimetres.

F lies on AB, D lies on BC and E lies on AC, as shown in Figure 1 below, so that $BDEF$ is a rectangle.

Denote lengths, in centimetres, $BF = a$, $AF = x$, $BD = c$ and $CD = y$.

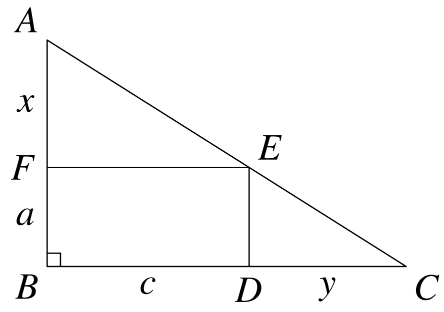


Figure 1

(a) Given that $a = 2$, $c = 6$ and $y = 3$, find x. (2 marks)

(b) It is now given that the area of rectangle $BDEF$ is 9 cm^2 and its dimensions are a whole number of centimetres.

Find all possible values for the length BC in centimetres. (8 marks)

Commentary

The first part of the question suggests that, given three of the values out of a, c, x, y, you should be able to find the fourth one. In the second part, you should expect to use the fact that all the lengths are whole numbers to figure out what three of the values can be.

In part (a), you may notice that $c = 2y$, so can you think of a reason why $x = 2a$? Remember that there are lots of right angle triangles in the diagram!

Although the question only asks you to fill in the table, the solution below is written as if full written explanations were required. The explanation at the start applies to both parts of the question.

Solution

Since FE is parallel to BC (because $BDEF$ is a rectangle), angles FEA and BCA are equal as corresponding angles. Triangles AFE and EDC are both right-angled and have another pair of equal angles. Hence they have three pairs of equals angles and so are similar. This means that their corresponding sides are in the same ratio. Because $BDEF$ is a rectangle, $FE = c$ and $ED = a$. Hence the ratio of the corresponding sides gives:

$$\frac{x}{c} = \frac{a}{y}$$

or, equivalently,

$$xy = ac.$$

(a) $x = \frac{ac}{y} = \frac{12}{3} = 4$

(b) The area of the rectangle $BDEF$ equals ac, so we know that $ac = xy = 9$. Since all four lengths are whole numbers, there are three options for each pair (a, c) and (x, y). The table below shows all nine possibilities for the four values, with the corresponding value of $BC = c + y$.

a	c	x	y	BC
1	9	1	9	18
1	9	9	1	10
1	9	3	3	12
9	1	1	9	10
9	1	9	1	2
9	1	3	3	4
3	3	1	9	12
3	3	9	1	4
3	3	3	3	6

The possible values for the length BC are 2, 4, 6, 10, 12 and 18.

2. This question requires full written explanations.

In Figure 2 below, each of the five circles A, B, C, D, E is to be coloured so that adjacent circles are different colours.

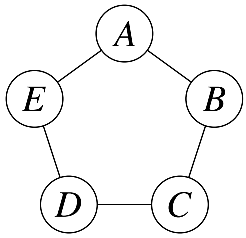


Figure 2

(a) In this part, circle A is to be coloured red and circle B yellow.

(i) Copy the diagram and show one possible way to colour the circles using colours red, yellow and green. Write 'R', 'Y' or 'G' in each circle to indicate its colour.

(ii) Explain why it is not possible to colour all five circles using only red and yellow. You should use labels A, B, C, D, E to refer to the circles.

(2 marks)

(b) In this part, the colours to be used are purple, orange and white. How many different ways are there to colour the circles using these three colours, so that adjacent circles are different colours? (8 marks)

Commentary

If you have never seen this sort of question before, it is a good idea to draw a few diagrams and think about what sorts of colourings are possible. You will hopefully soon realise that the best way to count them is to sort them into groups. For example, you could fix the colour of circle A, count how many ways there are to colour the rest, and then think about what happens when you change the colour of A.

In fact, the question is giving you a hint by telling you to fix the colours of both A and B.

For the first part, colour the circles C, D and E in order, making sure that each circle is a different colour from the previous one, and that E is not red. Note that this part only asks you to produce one example, but it should also give you some clues about what choices you have at each stage.

For the second part, with only two colours, you only have one choice at each stage, and you should be able to see why that doesn't lead to an allowed colouring.

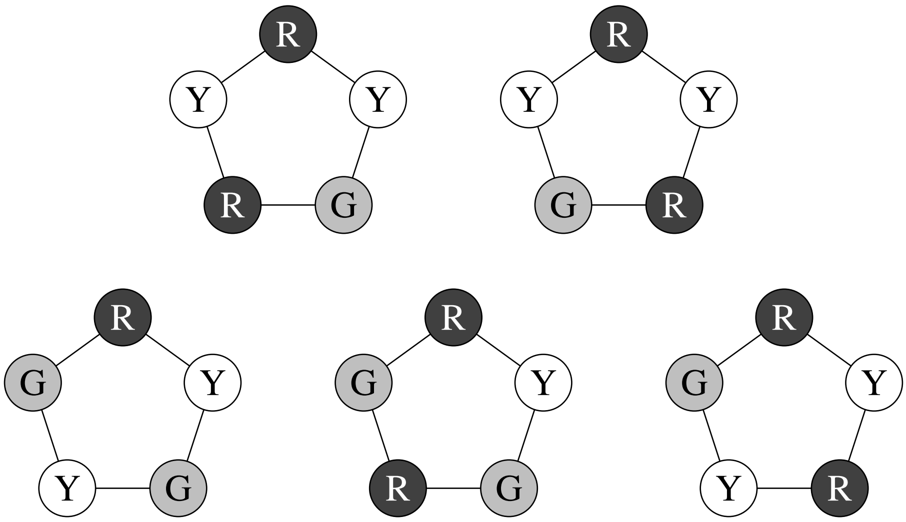
For the final part, you need to think about simplifying the counting by sorting different colourings into groups. This is where the first part is giving you a hint, by hopefully making you realise that, whatever combination of colours you pick for circles A and B, there will always be the same number of ways to colour the remaining three circles. So to get the final answer, you need to multiply this number by the number of ways to choose the colour combination for A and B.

The thought process that led you to construct an example in the first part should help you with your count. For example, you can think about colouring the circles in order (C, D, E) and considering what options you have at each stage.

There is an alternative approach to this question, which is to think about how often each of the three colours is used. This is described in the second solution to part (b) below.

SOLUTION

(a) (i) Any of the five colourings shown here is acceptable:



(ii) We already know that circle A is red and circle B yellow. This means that circle C has to be red, circle D yellow and circle E red. However, that makes E the same colour as A, so it is not possible to colour the circles using only those two colours.

(b) First of all, suppose that circle A is purple and circle B orange. There are then two options for circle C: it can be either purple or white.

If circle C is purple, circles D and E have to be orange and white in some order, giving two options.

If circle C is white, there are two options for D. If D is purple, E can be either orange or white. However, if D is white, E can only be orange. This gives three options with C white.

Hence, in the case when A is purple and B is orange, there are five options for the remaining three circles. (These correspond to the five options that were the possible answers to part (a)(i).)

But the situations will be exactly the same whatever the colours of circles A and B, each of those combinations giving five possible options for the other three circles. There are three options for the colour of A, and for each of them there are two options for the colour of B. This means that there are six possible ways to fix the colours of A and B, each of them resulting in five options for the colours of the other three circles.

Hence, the total number of possible colourings is $6 \times 5 = 30$.

Alternative solution for (b)

We are using three colours for five circles, so two of the colours must be used twice and the remaining colour only once. There are three choices for the colour which is used once, and five choices for the circle which is coloured with that colour. If we colour A with the "single" colour, there are only two options for the remaining four circles: B has to be the same colour as D, and C has to be the same colour as E. Hence the total number of options is $3 \times 5 \times 2 = 30$.

3. This question requires full written explanations.

(a) Figure 3 shows a semi-circle with centre O and diameter PQ and a point R on the semi-circle. By expressing other angles in the diagram in terms of p and q, prove that the angle in a semi-circle is $90°$. You must clearly state any geometrical facts you use.

(2 marks)

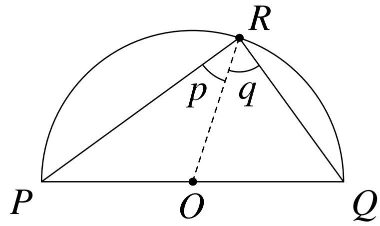


Figure 3

(b) Figure 4 shows an isosceles triangle ABC, with $AB = AC$. A semi-circle with centre M and diameter AC intersects AB at N.

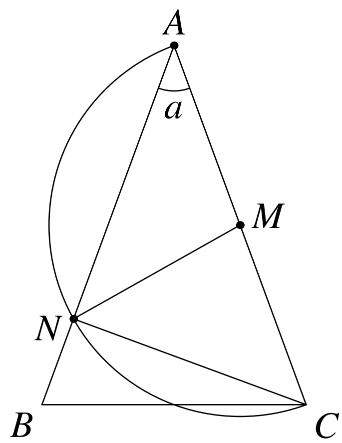


Figure 4

Given that $\angle AMN = 6\angle BCN$, find the size of angle $\angle BAC$, labelled a.

You must clearly state any geometrical facts you use. (8 marks)

Commentary

This question is about calculating angles, so you should expect to use angles in triangles and on straight lines. This is probably similar to angles questions you have met at school, except that none of the angle measurements are given. But you can use a strategy that is very common, not just in geometry, but in all of mathematical problem solving: Express as many angles as you can in terms of one or more variables and then form an equation to find their values. In this case, the question is helping you by defining some useful variables (p, q, a) for you.

Part (a) asks you to prove an important circle theorem that you may have already met at school. The key is deciding how to use the fact that P, Q and R are points on the circle with centre O. In the process of doing calculations, you may realise that it is not possible to find the sizes of angles p and q individually – they would change if the point R was moved to a different place on the semi-circle. However, the sum $p + q$ will always remain the same.

In part (b) you should expect to use the result you proved in part (a). Since you are given some information about angles AMN and BCN, it seems sensible to express both of them in terms of a and form an equation. Make sure you clearly state all the geometrical facts you use in your calculations, such as the sum of the angles in a triangle or on a straight line. You may find it helpful to annotate the diagrams with the angle calculations, but you should also explain them in writing.

In this question, all the angles in the diagram can be expressed in terms of a, but the solution below only calculates the ones that are needed to answer the question. You may well find yourself calculating all the angles while looking for the solutions; this is fine, but it is helpful to go back through your calculations at the end and reflect on ways you could make it shorter.

Note: The theorem in part (a) is considered a known theorem, and can be used without proof in Olympiad geometry questions. You just need to state clearly that you are using the fact that the angle in a semi-circle is 90°.

SOLUTION

(a) Since O is the centre of the semi-circle, $OP = OR$. This means that the triangle ORP is isosceles, so $\angle OPR = p$. Similarly, $\angle OQR = q$.

Angles in the triangle PQR add up to 180°, meaning that $2p + 2q = 180°$.

Hence, the required angle in the semi-circle, $\angle PRQ$, equals $p + q = 90°$.

(b) We are going to express angles AMN and BCN in terms of a.

Since M is the centre of the semi-circle, $MA = MN$ so triangle MAN is isosceles with $\angle MNA = a$. Using sum of angles in triangle MAN, $\angle AMN = 180° - 2a$.

Using the result from part (a) about angle in a semi-circle, we know that $\angle ANC = 90°$. But angles on a straight line add up to 180°, so $\angle BNC = 90°$ as well.

To find angle BCN we will use sum of angles in triangle BCN. We already know that $\angle BNC = 90°$. Using sum of angles in isosceles triangle ABC, we have

$$\angle NBC = \angle ABC = \angle ACB = \frac{180° - a}{2}.$$

Hence,

$$\angle BCN = 180° - 90° - \frac{180° - a}{2} = \frac{a}{2}.$$

The question says that $\angle AMN = 6\angle BCN$, so we can write an equation to find a:

$$180° - 2a = 6 \times \frac{a}{2} \implies 5a = 180° \implies a = 36°.$$

4. For this question, follow the instructions on the answer sheet.

(a) The equation $uv + 3u - 3v - 18 = 0$ is equivalent to $(u - a)(v + b) = 9$. Write down the values of a and b. (1 mark)

(b) Hence find all integer solutions of the equation $uv + 3u - 3v - 18 = 0$.

(3 marks)

(Note: 'integers' means whole numbers, which can be positive, negative or zero.)

(c) Find all integer solutions of the equation $x^2 - y^2 + 6y - 18 = 0$.

(6 marks)

COMMENTARY

This question is about solving equations by factorising, as hinted at in part (a). You are probably familiar with solving quadratic equations by factorising, and know that you need to make a factorised expression equal to zero. The same idea can be applied to any expression that can be factorised. For example, the equation $xy + 3x - 3y - 9 = 0$ can be written in factorised form as $(x - 3)(y + 3) = 0$ and so the possible solutions are $x = 3$ (with y being any real number), or $y = -3$ (with x being any real number).

If the factorised expression were not equal to zero, for example if we were trying to solve $xy + 3x - 3y - 9 = 9$, then (almost) any real value of x would have a corresponding value of y. The equation could be written as $(x - 3)(y + 3) = 9$, so we could take any $x \neq 3$ and find the corresponding $y = \frac{9}{(x-3)} - 3$.

However, if we are only interested in integer (whole number) solutions, then not only do we need to pick an integer value for x but we also need to make sure that the corresponding value of y is an integer. Looking at the factorised form of the equation, we can see that the two brackets need to be factor pairs of 9. To ensure we find all the solutions, we need to remember that the two factors can be either way round, as well as include negative factors. For part (b), listing all the factor pairs in a table is a good way to make sure you don't miss out any solutions.

Part (c) can be done in several ways; the one we show here is making a link to part (b) and using the solutions found already. The difference of two squares, $x^2 - y^2$, is an important expression which you could recognise. Factorising it might suggest to try replacing $x + y$ and $x - y$ with u and v in some order – see whether you can get $6y$ to equal $3u - 3v$.

SOLUTION

(a) Expanding the brackets and rearranging, we get:

$$uv + bu - av - (ab + 9) = 0.$$

Hence, $a = b = 3$.

(b) Since u and v are integers, the two brackets $(u - 3)$ and $(v + 3)$ need to be factors of 9.

The table shows all the factors pairs of 9 and the corresponding values of u and v.

$u - 3$	$v + 3$	u	v
1	9	4	6
9	1	12	−2
−1	−9	2	−12
−9	−1	−6	−4
3	3	6	0
−3	−3	0	−6

(c) Let $u = x + y$ and $v = x - y$. Then $uv = x^2 - y^2$ and $3u - 3v = 6y$. Hence the equation is equivalent to the equation from part (b), and the solutions can be found by setting $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$ and checking that they are all integers.

The table shows the values of u and v found in part (b) and the corresponding values of x and y.

u	v	$x = \frac{u+v}{2}$	$y = \frac{u-v}{2}$
4	6	5	−1
12	−2	5	7
2	−12	−5	7
−6	−4	−5	−1
6	0	3	3
0	−6	−3	3

5. This question requires full written explanations.

The positive integers from 1 to N are equally spaced around a circle in such a way that the sum of any two neighbouring numbers is equal to the sum of the two numbers diametrically opposite them.

(a) Figure 5 shows an example with $N = 6$. Number 1 is diametrically opposite number 4 and number 2 is diametrically opposite a.

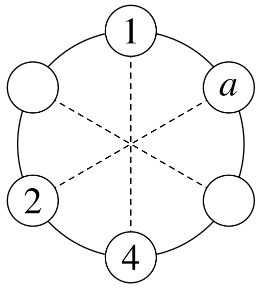


Figure 5

Copy and complete the diagram to show one possible way of arranging numbers 1 to 6. No explanation is needed in this part. (1 mark)

In the rest of this question, $N = 222$.

(b) Show that the difference between any pair of diametrically opposite numbers is the same. You may find it helpful to refer to Figure 6 in your explanation. (1 mark)

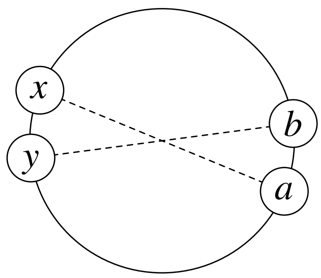


Figure 6

(c) Suppose that there is an arrangement in which number 41 is opposite number 1. By considering which numbers would need to be placed opposite 40, 120 and 200, show that such an arrangement is not possible. (1 mark)

(d) Find all possible numbers which could be placed opposite number 1. You need to explain why those are the only possibilities. In each case, you also need to explain why it is possible to arrange the remaining numbers. (7 marks)

Commentary

This is a really difficult problem with lots of things to do. The first part already asks you to produce one example with a small value of N, and you may want to try a few more, for example $N = 8$ and $N = 10$, to get a feel for the problem.

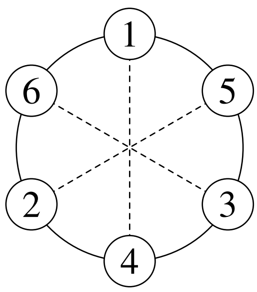
You may realise that the problem can be split into two parts: deciding how to pair up the numbers which should be diametrically opposite each other, and then arranging them around the circle. Parts (b) and (c) give you some help with the first task; trying some examples should help you with the second.

The observation in part (b) is important because it means that, once you have fixed the number which goes opposite 1, the options for what goes opposite any other numbers become very restricted. For example, if 7 is opposite 1, what numbers could possibly go opposite 20? Then thinking about part (c) hopefully leads you to realise that, in fact, only one of those options is possible. This means that, having decided what goes opposite 1, you can pair the other numbers, one by one. But then part (c) helps you realise what can go wrong when you try to do this.

Once you have understood how parts (b) and (c) are helpful, you may be ready to solve the final part of the problem. If not, go back to some examples. You may find that slightly larger numbers are more helpful now; maybe try $N = 18$ or $N = 22$. The question you are asking is: which numbers can I pair up with 1 so that the remaining numbers can be paired up correctly?

The final part of the solution needs to explain why, once the numbers have been paired up correctly, they can be arranged around the circle in the required way. To see what can go wrong, try arranging 1 to 8 with the pairing $(1, 2)$, $(3, 4)$, $(5, 6)$, $(7, 8)$ and also with $(1, 5)$, $(2, 6)$, $(3, 7)$, $(4, 8)$. You then need to explain why, with 222 numbers, this problem does not arise.

(a) With the given starting numbers, there is only one way to complete the diagram: $2+4 = 1+a$, so $a = 5$. The remaining numbers, 5 and 6, can only be placed in the way shown here:



(b) In the diagram given in the question, $x + y = a + b$, so $x - a = b - y$. This applies to any four numbers in a similar configuration, so the difference between the diametrically opposite numbers is always the same.

(c) Using the result from part (b), the difference between any pair of diametrically opposite numbers is 40. Therefore the number opposite 40 has to be 80, the number opposite 120 has to be 160 (since 80 has already been used), and the number opposite 200 has to be 240 (since 160 has already been used). But 240 is too big, so an arrangement with the difference of 40 is impossible.

(d) Suppose that there is an arrangement in which the number opposite 1 is $d + 1$ so that the difference between diametrically opposite numbers is d. We are going to show that $2d$ must be a factor of 222.

Since the number opposite 1 is $d + 1$, we know that $d + 1 \leq 222$. Using the fact that the difference between each pair of opposite numbers is d, we can see that the numbers opposite $1, 2, \ldots, d$ must be $d + 1, d + 2, \ldots, 2d$, respectively. This creates a "block" of $2d$ numbers.

If $2d+1$ is still smaller than or equal to 222, the number opposite it must be $3d+1$, because $d + 1$ has already been used. Furthermore, this means that $3d + 1 \leq 222$ so all numbers from $2d + 1$ to $3d$ must be included, and the numbers opposite them must be $3d + 1$ to $4d$ (as the numbers from $d + 1$ to $2d$ have already been used). This creates another "block" of $2d$ numbers, from $2d + 1$ to $4d$.

We can continue with the same reasoning to conclude that the numbers come in blocks of $2d$, where numbers $2kd + 1, 2kd + 2, \ldots, 2kd + d = (2k + 1)d$ are opposite the numbers $(2k + 1)d + 1, (2k + 1)d + 2, \ldots, (2k + 1)d + d = 2(k + 1)d$. Hence $2d$ must be a factor of 222.

The even factors of 222 are $2, 6, 74, 222$ and so d could be $1, 3, 37$ or 111. Hence, the numbers that can be opposite number 1 are $2, 4, 38$ and 112.

We still need to show that each of the situations described above is possible, by constructing an example with each of the above values of d. First, we pair up the numbers according to the "blocks" described above. To arrange them around the circle, we note the diagram from part (b), in which $x - a = b - y$. So, if x is the smaller of the two numbers in a pair (so $x < a$) then y needs to be the larger of the two numbers in its pair (so $y > b$). For example, in the case when $d = 37$, 1 is opposite 38; we can place 2 next to 38 and 39 next to 1.

So we arrange the numbers around the circle as follows. Start with 1 and $d + 1$. Then pick another pair and, moving anti-clockwise, place the larger number next to 1 and the smaller number next to $d + 1$. Then pick another pair and place the new larger number next to the previous smaller number. Continue like this until all the pairs have been used up. There are 111 pairs, which is an odd number. This means that, with the final pair, the smaller number will be next to $d + 1$ and the larger number next to 1, which is correct. For all four possible values of d, an arrangement formed following this method works.