UK Maths Trust

Mathematical Olympiad for Girls

supported by Jane Street Overleaf

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. They are not intended to be the 'best' possible solutions; in some cases we have suggested alternatives, but readers may come up with other equally good ideas.

Full explanations are given even for the answer-only questions; it is hoped these are helpful if you are preparing for other olympiads. All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or or if there are small gaps in your explanation, then you will normally receive nearly all the marks.

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1. *This question requires answers only.*

Consider the number $M = 99...99$ which consists of several digit nines. A single division sign is placed between two adjacent digits of M and the resulting calculation is evaluated to produce a whole number N.

(a) In the case when M has nine digits,

(i) How many possible values can N take? [1 mark]

(ii) How many digits does the smallest possible value of N have? [2 marks]

(b) In the case when M has 2025 digits,

(i) How many possible values can N take? [3 marks]

(ii) How many digits does the smallest possible value of N have? [4 marks]

COMMENTARY

In a question like this it is sensible to start by looking at some small cases, which is what part (a) is encouraging you to do. You can look at examples with even fewer nines to develop and test your ideas.

The key is to come up with a simple way to do the division.

For example, if we want to divide a number with six nines by a number with two nines, we can write the first number as $990000 + 9900 + 99$, so dividing it by 99 gives $10000 + 100 + 1 = 10101$. On the other hand, if the first number has five nines, it equals $99000 + 990 + 9$, which is not a multiple of 99.

Trying some examples like this will hopefully lead you to an idea you can apply in both parts of the question. You should note both when the division gives a whole number, and what the form of the answer is.

Write the division as $A \div B$, where A consists of a nines and B consists of b nines. We are going to show that the calculation produces a whole number if, and only if, a is a multiple of b. We are then going to apply that idea to both parts of the question.

We must have $A \geq B$ for the division to produce a whole number so $a \geq b$. Write $a = kb + r$, where $k \geq 1$ and $0 \leq r < b$. This means that we can split A into k blocks of b nines, going from left to right, with r nines left over.

Each block of b nines represents a number which is B followed by some zeros. It is therefore divisible by B. The remaining r nines represent a number which is smaller than B, so dividing A by B leaves a remainder which is a number with r nines.

Therefore, the only way that $A \div B$ can be a whole number is if $r = 0$, in which case a is a multiple of b.

(a) The possible ways to write 9 as $a + b$ with $a \geq b \geq 1$ are $8 + 1$, $7 + 2$, $6 + 3$ and $5 + 4$. Out of those, a is a multiple of b only in the cases $a = 8, b = 1$ and $a = 6, b = 3$. Therefore N can take two possible values.

The smallest possible value of N is when a and b are closest together since this minimises A and maximises B. In this case, it is $999999 \div 999 = 1001$, which has 4 digits.

(b) We need to write $2025 = a + b$ where $a \geq b \geq 1$ and a is a multiple of b. Each of these pairs of a and b will give a different value of the whole number N, with the smallest N corresponding to the pair in which a and b are closest together, meaning that b is as large as possible.

Writing $a = kb$, with $k \geq 1$, we have $2025 = kb + b = (k+1)b$. This means that $(k+1)$ and b are factors of 2025 with $(k+1) \geq 2$.

We know that 2025 has 15 factors. However, one of them is 1, which is not a possible value of $(k+1)$. Hence, there are 14 possible values of N.

The smallest value of N occurs when b is as large as possible. This is the case when $(k+1) = 3$, $b = 675$, resulting in $a = kb = 1350$. In the division $A \div B$, A consists of two blocks of 675 nines, so N has 676 digits (a 1 followed by 674 zeros followed by another 1).

Note

The proof that b needs to divide a can also be written using algebra, instead of thinking about blocks of nines.

Note that the number with a nines can be written as $10^a - 1$. It is well known that $x^k - 1$ has a factor of $x - 1$ for all positive integers k.

If $a = kb + r$, then

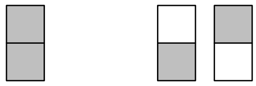
$$\begin{aligned} 10^a - 1 &= 10^{kb} \times 10^r - 1 \\ &= 10^r(10^{kb} - 1) + (10^r - 1) \\ &= 10^r((10^b)^k - 1) + (10^r - 1) \end{aligned}$$

The first term has a factor of $10^b - 1$, which is our number B. Since $r < b$, the second term is smaller than B, so that is the remainder of the division and equals zero if, and only if, $r = 0$.

The full result mentioned above says that $x^k - 1 = (x-1)(x^{k-1} + x^{k-2} + \cdots + x + 1)$, and this tells you the form of the resulting integer N.

2. *This question requires full written solutions.*

Tom has a large supply of two types of dominoes, Type A and Type B. Type B dominoes can be rotated 180° so that the grey square is on top.



type A type B

Tom wants to select three dominoes and place them next to each other to create a 2×3 rectangle (so the dominoes remain vertical, as shown above). He wants both top and bottom rows of his rectangle to contain at least one square of each colour.

(a) (i) How many sequences can Tom make which contain exactly one Type A domino? [2 marks]

(ii) How many sequences of three dominoes can he make in total? [2 marks]

(b) Each of the white squares on Type B dominoes has a whole number between 1 and 6 (inclusive) written on it. Grey squares have no numbers on them. There are several copies of each numbered Type B domino.

(The numbers do not change when a Type B domino is rotated; for example a 6 does *not* become a 9.)

Tom wants to select and arrange three dominoes as before (with both top and bottom rows containing at least one square of each colour, and using dominoes of either type), but now he also wants the numbers on the top row to add up to 6 and the numbers on the bottom row to add up to 6.

In how many ways can he do this? [6 marks]

COMMENTARY

The key to counting questions is to organise your work systematically, making sure you consider all possible cases and do not include any case more than once.

If there are not too many options it is fine to list them all, as long as you explain how you have ensured that you have not missed any. It is also helpful to realise when you can group the options in some way, and then only list some of the groups. For example, in part (a)(i), there are three positions where the Type A domino can go. So you can put it in the first position, list all possible options for the other two dominoes, and then multiply that number by three. (Note that, in the solution below, we have taken a different approach.)

The structure of the question strongly suggests that for sub-part (a)(ii) you should consider two separate cases: one including a Type A domino and one without it. Note that you need to explain why those are the only cases (i.e. why you can't have more than one Type A domino).

Part (b) can be linked to part (a). For each arrangement from part (a), think about how you can number the white squares in order to satisfy the conditions.

(a) (i) When using a Type A domino, Tom must also use two Type B dominoes, one in each orientation, in order to have one white square in each row. He is therefore using three different-looking dominoes, and they can be arranged in $3 \times 2 \times 1 = 6$ ways.

(ii) Call the two orientations of the Type B domino (as shown in the diagram) B_1 and B_2. Tom must use at least one of each of these two types, otherwise one of the rows would contain only grey squares.

This means that Tom can use the following sets of dominoes: one of each type; one B_1 and two B_2s; or one B_2 and two B_1s.

The first case gives the six sequences described in part (i). The second and the third case give the same number of sequences, so we only need to look at the second case.

The single B_1 domino can go in any of the three places, with the remaining places being take up by the two B_2 dominoes. Hence there are three sequences for the second case, and another three for the third case.

This gives the total of $6 + 3 + 3 = 12$ possible sequences.

(b) We consider the three cases described in the previous part. For each possible sequence of the dominoes, we count how many ways there are to number them.

In the case with one of each type of domino, each row contains only one white square, so they must both have the number 6 written on them. There is only one possible numbering for each of the six sequences, giving six possible ways for this case.

In the case with one B_1 and two B_2 dominoes, the B_1 domino must have the number 6 on it, but the two B_2 dominoes can have one of the following five pairs: $(1, 5)$, $(2, 4)$, $(3, 3)$, $(4, 2)$ or $(5, 1)$. So for each of the three sequences of dominoes, there are five ways to select the numbers. This gives $3 \times 5 = 15$ possibilities for the second case.

The third case is the same as the second case. So the total number of possibilities is $6 + 15 + 15 = 36$.

3. *This question requires answers only.*

(a) Positive (non-zero) whole numbers a and b satisfy $(a+b)(a-b) = 45$. Find all possible values of a. [2 marks]

(b) Priya and Rhia each create a sequence of positive integers.

Priya starts with 1000 and adds consecutive odd numbers, so that her sequence begins 1000, 1001, 1004, 1009.

Rhia starts with 3025 and also adds consecutive odd numbers.

(i) How many numbers (including 3025) appear in both sequences? [4 marks]

(ii) Find the **second** smallest number that appears in both sequences. [4 marks]

COMMENTARY

This is another question where part (a) gives you a strong hint for how to approach part (b): you are looking to write an equation which you can factorise, and then look for factors of some number.

You may already know that consecutive odd numbers add up to a square number (for example, $1 + 3 + 5 + 7 = 16$). Even if you don't, you should be able to spot this pattern by continuing Priya's sequence for a few more terms. This is an example of a known result that could be stated without proof, even in a full solution question.

Note that in the solution below we use two different letters, n and k, for the general terms of the two sequences. A common mistake is to use the same letter, but that would mean that the equal terms are always in the same position, which is clearly not the case.

(a) Both $(a+b)$ and $(a-b)$ must be factors of 45. Furthermore, as b is positive, $(a+b)$ must be the larger factor. We therefore have the following possibilities:

$a+b$	$a-b$	a
45	1	23
15	3	9
9	5	7

The possible values of a are 7, 9 and 23.

(b) Adding consecutive odd numbers, starting from 1, gives consecutive square numbers. Therefore the nth term of Priya's sequences is $1000 + n^2$ and the kth term of Rhia's sequence is $3025 + k^2$, where $n, k \geq 0$ (the "0th term" corresponds to the starting number).

(i) For a number to appear in both sequences, we need

$$\begin{aligned} & 1000 + n^2 = 3025 + k^2 \\ \iff & n^2 - k^2 = 2025 \\ \iff & (n-k)(n+k) = 2025 \end{aligned}$$

This shows that both $(n - k)$ and $(n + k)$ need to be factors of 2025. Since $k \geq 0$, we must have $n + k \geq n - k$. Note that we need not consider negative factors because n and k are both positive, so it is not possible for both $n + k$ and $n - k$ to be negative.

Note that $n - k = n + k = 45$ is allowed, as this corresponds to $k = 0, n = 45$, giving $1000 + 45^2 = 3025 + 0^2 = 3025$ as the smallest number which appears in both sequences.

The solutions therefore correspond to the factor pairs of 2025, with $n + k$ being the larger one. Note that 2025 is odd, giving $n - k$, $n + k$ both odd, so n and k are indeed both integers for each factor pair.

Since 2025 has fifteen factors, it has eight factor pairs, including 45×45. Hence there are eight numbers which appear in both sequences.

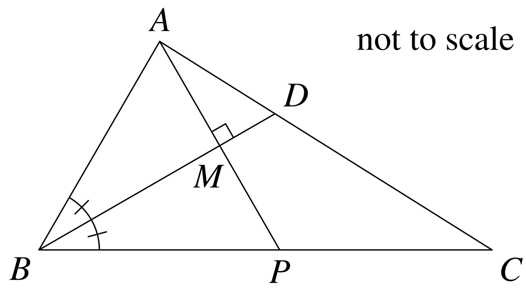
(ii) Smaller numbers in the sequence correspond to the smaller values of k. The value of k will be the smallest when the two factors of 2025 are closest together.

When the two factors are both 45, the value of k is 0, giving 3025 as the smallest number that appears in both sequences. The next factor pair where the factors are closest to each other is 27×75. This gives $n - k = 27$ and $n + k = 75$ so $k = 24$.

The second smallest number that appears in both sequences is $3025 + 24^2 = 3601$.

4. *This question requires full written solutions.*

The diagram shows triangle ABC with side lengths $AB = 85$, $BC = 160$ and $CA = 103$ units. The bisector of angle ABC intersects the side AC at point D. The line through A perpendicular to BD intersects BD at M and BC at P.



(a) Prove that M is the midpoint of AP. [2 marks]

(b) Find the length of PC, justifying your answer. [1 mark]

(c) The line through A perpendicular to the bisector of angle ACB intersects it at N. Find the length of MN. [7 marks]

Commentary

This is a geometry problem where each part builds on the previous one.

In part (a), the goal is to show lengths AM and MP are equal. Since triangles AMB and PMB share a side and have a right angle at M, it is natural to try to prove the two triangles are congruent. There are several congruence rules (ASA, SAS, SSS, RHS), so the task is to identify the one that is most useful here (in fact, multiple work).

For part (b), it helps to mark the given lengths on the diagram. We already know length BC, so finding length BP would allow us to work out length PC. This connects back to some of the work we did in part (a).

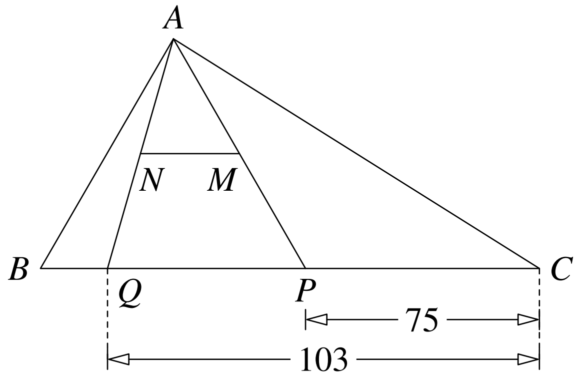
In the final part, the problem introduces a similar construction on the other side of the triangle. The results from parts (a) and (b) already give useful information about the diagram. The final step is to find a way to link the new segment MN to lengths we can compute using what we have established earlier.

In may be helpful to redraw the diagram to show only the information relevant to part (c), as we did in the solution below.

(a) Since the line BM bisects angle PBA, angles PBM and ABM are equal. Since BM is perpendicular to AP, angles BMP and BMA both equal 90°. Therefore, triangles BPM and BAM have two equal angles and share the side BM, so they are congruent (ASA). This means that $MP = MA$, so M is the midpoint of AP.

(b) From the congruence of triangles BPM and BAM it also follows that $BP = BA = 85$. Hence $PC = BC - BP = 75$.

(c) Let the line AN intersect BC at Q. By the same argument as above, N is the midpoint of AQ and $CQ = CA = 103$. We can then draw the following diagram:



In triangle AQP, N and M are the midpoints of the sides AQ and AP. Therefore, the triangles AMN and APQ are similar (SAS), with AMN being half the size of APQ. This means that the length MN is half the length QP.

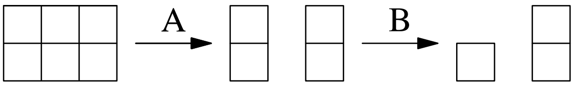
We can calculate $PQ = CQ - PC = 103 - 75 = 28$ so $MN = 14$.

5. *This question requires full written solutions.*

Two players, A and B, play a game with a set of square tiles. To start, the tiles are arranged in a rectangle. A turn consists of removing a row or column of tiles from one rectangle (possibly splitting it into two). A rectangle consisting of only one row or one column may be removed in a single turn.

Players take turns, with Player A going first. The player removing the final tile(s) wins.

(a) In this part, the game starts with a 2×3 rectangle. The diagram below shows the first two moves. Player A starts by removing the middle column.



Complete the diagram to show one way in which Player A can win the game.

[1 mark]

(b) In this part, the game starts with a 20×25 rectangle. Describe a strategy for Player A which guarantees that they will win the game, no matter what Player B does on their turns. [2 marks]

(c) In this part, the game starts with a 20×24 rectangle. Determine which player has a winning strategy. [7 marks]

You must describe what the winning player should do on each turn, which may depend on what the other player does, and explain why these moves always lead to a win.

COMMENTARY

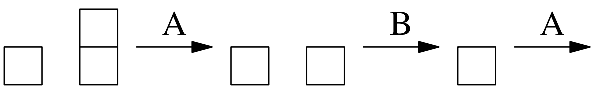
The first step in solving a problem like this is to understand the rules of the game. Part (a) shows you one example of how it works, and also hopefully suggests what a useful starting move might be – at least in some situations.

Try playing the game with similar examples first. For example, keep three columns and try changing the number of rows. Make sure A can win whatever B chooses to do for their moves (including if they select a move which you think is not "the best"). This may lead you to the solution to part (b).

Then think what happens if there are only two columns instead of three. In what situations can A still apply the same strategy? In the cases where they can't, can similar ideas still be used to determine who wins?

In the first solution below, part (c) uses what was discovered in part (b). The alternative solution presents a different strategy which can be applied to both parts.

(a) We show the tiles starting from the end of Player B's first move:



…and so A wins

(b) Player A should start by removing the middle column, leaving two 20×12 rectangles. Let us colour one of those rectangles blue and the other one red. Since the two rectangles are identical, each red tile has a corresponding blue tile.

In subsequent moves, whenever Player B removes a blue row or column, Player A should remove the corresponding red row or column, and vice versa. If Player A follows this strategy, the red and the blue tiles will be in identical arrangements after each of Player A's turns. Since the number of tiles decreases every turn, Player B will eventually remove the final tile(s) of one of the colours, and then Player A can remove the remaining tile(s) of the other colour, thus winning the game.

(c) Player B has a winning strategy.

Call a rectangle "even" if both its length and width are even, and "odd" otherwise, so that an "odd" rectangle has at least one odd side. The game starts with an even rectangle.

On their first move, Player A has two options: either remove one of the edges, leaving an odd rectangle, or split the rectangle in two, leaving one even and one odd rectangle. Either way, after A's first move, there will be one odd rectangle.

Let us colour the odd rectangle green and the even rectangle (if any) white. On their first move, Player B should remove the middle row or column of the odd (green) rectangle, splitting it into two identical rectangles (or removing the whole odd rectangle if it is a single row or column to begin with). Player B will then choose their moves depending on whether Player A removes green or white tiles.

Whenever Player A removes some green tiles, Player B should remove corresponding green tiles, as in part (b). Whenever Player A removes some white tiles, they will be creating an odd rectangle (and possibly leaving another even rectangle). Player B's response should be to colour the odd rectangle green and remove its middle row or column (which could mean removing the entire rectangle if it consists of a single row/column).

In this way, at the end of Player B's move, white tiles will always form even rectangles and green tiles will always form pairs of identical rectangles. So whatever Player A does on their next move, Player B will always be able to choose their move as described above.

Furthermore, Player A cannot have the final move. If there are only white tiles left at the start of their turn, the rectangle is even meaning that it has at least 2 rows and 2 columns and cannot be removed in a single move. If there are any green tiles left, they will form (at least) one pair of separate identical rectangles, which can also not be removed in a single move.

Since the tiles are being take away every turn the game must eventually end. Since A cannot have the final move, B is guaranteed to win if they follow the above strategy.

ALTERNATIVE

In this variation of the solution to part (c), Player B does not mirror Player A, but instead copies Player A in another way. Player B considers the file (row or column) which Player A just removed, and removes the file which is the rotation through $180°$ of Player A's file about the centre of the original rectangle.

This file is always available for removal, and the resulting figure in invariant under a 180° rotation about the centre. Thus Player A is forced to break this rotational symmetry, and Player B responds by restoring it. In the end the figure is empty, and that is symmetric under rotational symmetry, so Player B wins.

In fact this method can also be used to show that Player A wins in part (b). Player A begins by removing a central file (which only exists in the case of an odd rectangle). The resulting figure is symmetric under a 180° rotation about the centre of the original rectangle, and any move that Player B makes will break this rotational symmetry. Then Player A can restore the rotational symmetry by rotational copying. Once again, the game must end (as each move removes at least one tile) and the empty figure is rotationally symmetric, so Player A wins.