UK Maths Trust

Mathematical Olympiad for Girls

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. They are not intended to be the 'best' possible solutions; in some cases we have suggested alternatives, but readers may come up with other equally good ideas.

All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or been unclear in your explanation, then you will normally receive nearly all the marks.

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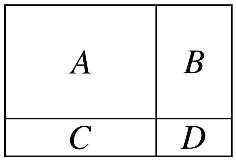
1. *This question requires answers only.*

A rectangular apartment is divided into four rectangular rooms, by two straight interior walls which are parallel to the exterior walls. The rooms have areas A, B, C and D metres squared.



(a) Find the value of D when $A = 8$, $B = 4$ and $C = 12$. [1 mark]

(b) Find the value of C when $A = 7.5$, $B = 6.5$ and $D = 26$. [2 marks]

(c) Find an expression for C in terms of A, B and D. [2 marks]

(d) In this part of the question, A, B, C and D have the values found in part (a). Additionally, the length and the width of each room are whole numbers.

Let P be the perimeter of the apartment. Find all possible values of P. [5 marks]

Commentary

You could write some equations to find the side lengths of each room. However, for the first three parts, it is simpler to consider ratios of areas.

Rooms A and B have the same width, so their areas are in the same ratio as their length. The same is true for rooms C and D. It follows that $A : B = C : D$.

(a) $8 : 4 = 12 : D$, so $D = 6$.

(b) $7.5 : 6.5 = C : 26$ so $C = 30$.

(c) $C = \dfrac{AD}{B}$.

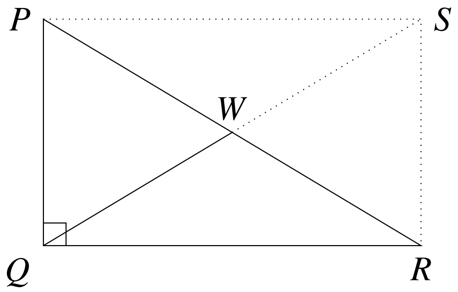
(d) Denote the lengths of A and B by x and y, and the widths of A and C by z and t, respectively. Then the area of B gives $yz = 4$, so there are three options: $(y, z) = (4, 1), (2, 2)$ or $(1, 4)$.

The first case gives $t = 1.5$ which is impossible.

In the second case, $x = 4, t = 3$ so $P = 2(6 + 5) = 22$.

In the third case, $x = 2, t = 6$ so $P = 2(3 + 10) = 26$.

2. (a) Triangle PQR has a right angle at Q and W is the midpoint of the side PR. Use the diagram below to show that $QW = RW$. [2 marks]



(b) Triangle ABC has a right angle at B and $AB < BC$. Let M be the midpoint of AC. Let D be the reflection of C in the line BM (extended beyond M).

Given that angle DAC is $x°$ find, in terms of x, the size of angle BAC. [8 marks]

You must show all your calculations and say what geometrical result(s) you used at each step.

COMMENTARY

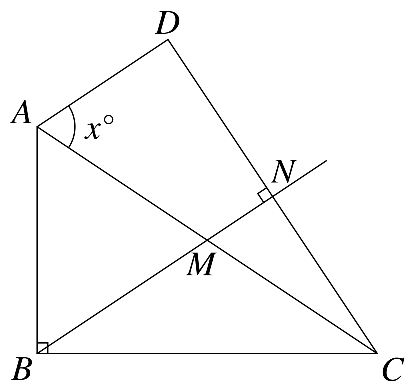
Start by drawing a clear diagram and identify the angle you know and the angle you want to find.

Angle BAC can be found from triangle ABM, and part (a) tells us something important about it. You are given angle DAC; what can you say about the line AD? Think about how you can sue properties of reflection.

(a) The diagonals of a rectangle are equal and bisect each other. Hence $QW = \frac{1}{2}QS = \frac{1}{2}PR = RW$.

(b) Let N be the intersection of CD and BM. By the properties of reflection, N is the midpoint of BD and BNC is a right angle.

In triangle ACD, M and N are midpoints of AC and CD, so MN is parallel to AD. It follows that ADC is a right angle and angle $NMC = x°$.



Using the result of part (a), $BM = MC = MA$ so triangle AMB is isosceles with angle angle $x°$ at M. Since angles in a triangle add up to 180°, we find that angle MCB equals $\left(\frac{180 - x}{2}\right)^{\circ}$.

3. *This question requires answers only.*

(a) You have three red and four blue cubes. The cubes of the same colour are identical.

This part of the question is about the number of ways to distribute all seven cubes among some boxes, following these rules:

- A box may be empty (so it is possible to put all seven cubes in one box).
- The boxes are considered different (so putting all seven cubes in the first box counts as a different solution from putting all seven cubes in the second box).

(i) How many ways are there to distribute all seven cubes among two boxes? [2 marks]

(ii) How many ways are there to distribute all seven cubes among three boxes, so that no colour appears in all three boxes? [3 marks]

(b) The *factorial* of a positive integer n is defined as $n! = 1 \times 2 \times 3 \times \cdots \times n$. For example, $3! = 1 \times 2 \times 3 = 6$.

In how many ways can we write $10! = abc$, where a, b, c are positive integers, such that there is no factor greater than one which is common to all three of them?

Reorderings of a, b, c count as different solutions. For example, $120 \times 54 \times 560$ is not allowed because all three numbers share a common factor of 2, while $640 \times 162 \times 35$ is allowed and counts as a different solution from $162 \times 640 \times 35$. [5 marks]

COMMENTARY

In part (a), the most important thing to realise is that the two colours can be distributed independently. When you have three boxes you need to think carefully how you are going to ensure that only one or two boxes contain each colour.

Before starting part (b), think about how part (a) can help. You are trying to write a number as a product, so you are trying to distribute all the factors you have among the "boxes" a, b and c. You should find that you can count in exactly the same way as in part (a).

(a) We count the number of ways to distribute the cubes of each colour separately. Those numbers then need to be multiplied because for every distribution of red cubes we can have every possible distribution of blue cubes.

(i) The three red cubes can be split as $0 + 3, 1 + 2, 2 + 1$ or $3 + 0$, which is 4 ways. Similarly, the blue cubes can be split in five ways. Hence the total number of ways to distribute the cubes is $4 \times 5 = 20$.

(ii) There are 3 ways for all the red cubes to be in the same box. Otherwise one box contains no red cubes, and there are 2 ways to distribute them between the other two boxes $(1 + 2$ or $2 + 1)$. There are three ways to select the box with no red cubes. Hence the number of ways to distribute the red cubes is $3 + (3 \times 2) = 9$.

Similarly, the number of ways to distribute the blue cubes is $3 + (3 \times 3) = 12$.

Hence the total number of ways to distribute the cubes is $9 \times 12 = 108$.

(b) We can factorise $10! = 2^8 \times 3^4 \times 5^2 \times 7$. Each of the numbers a, b and c is a product of some powers of $2, 3, 5$ and 7, where some of the powers must be zero to avoid all three numbers having a common factor greater than one.

The factor of 7 can be in exactly one of the three numbers, giving three options.

Either one of the three numbers is a multiple of 25, or two of them are multiples of 5, giving $3 + 3 = 6$ options.

Distributing the factors of 3 among the three numbers is the same as distributing the four blue cubes in part (a)(ii), which is 12 options.

Finally, we can find the way to distribute the factors of 2 in the same way (imagine there are eight orange cubes), giving $3 + (3 \times 7) = 24$ options.

Therefore the total number of choices for the values of a, b and c is $3 \times 6 \times 12 \times 24 = 5184$.

4. This question is about finding winning strategies in two-player games. We say that a player has a *winning strategy* if they can win the game no matter what moves the other player makes.

(a) In the first game, Players A and B start with a pile of 100 counters and take turns to remove either 2 or 3 counters from the pile. The winner is the player who removes the final counter. If only one counter remains, the result is a draw.

Player A goes first. Describe the winning strategy for Player B, and explain why this strategy guarantees a win. [2 marks]

(b) Players A and B are now playing a different game. They start with a pile of 700 counters, and take turns to remove 1, 3 or 4 counters from the pile. The winner is the player who removes the final counter. (If there are only two counters left, the current player must remove one counter.)

Player A goes first.

(i) Show that Player B can ensure that, at the end of their first turn, there are either 693 or 695 counters remaining. [1 mark]

(ii) Show that Player B can win if, at the end of one of their turns, there are 7 or 9 counters remaining. [1 mark]

(iii) Determine which player has a winning strategy.

You should

- Describe how the winning player should select their moves, which may depend on the other player's moves.
- Explain why this strategy guarantees that this player will take the final counter.

[6 marks]

COMMENTARY

The best way to tackle problems like this is to try playing the game. It is useful to think "from the end": which numbers of counters would the winning player reach on their penultimate move?

These "winning positions" should not be too difficult to find for the first game. In the second game, parts (i) and (ii) suggest that there are two types of winning positions.

When answering the final part, you need to make sure that you

- Describe how the winning player should choose their moves;
- Explain why they are always able to make those moves;
- Explain why those moves mean that the other player cannot reach zero counters.
- Conclude that the winning player must eventually reach zero counters.

(a) Player B should do make the opposite move from Player A: whenever Player A removes two counters, Player B should remove three, and vice versa.

This means that the number of counters decreases by 5 every two turns. Since it starts as a multiple of 5, it will be a multiple of 5 at the end of each of Player B's turns, and will never be a a multiple of 5 at the end of Player A's turn.

Since the number of counters decreases every turn, it will eventually reach zero, which is a multiple of 5, so this must happen at the end of Player B's turn.

(b) (i) Player B can achieve this as follows. If Player A removes one counter, Player B should remove 4 to reach 695. If Player A removes three or four counters, Player B should remove four or three, respectively, to reach 693.

(ii) Note that if Player B ends their turn with 2 counters remaining, they are guaranteed to win (because Player A must remove 1 and then Player B takes the final counter.).

Starting with 7 counters, if Player A removes four or three, Player B can reach zero. And if Player A removes one, Player B can remove four to reach 2.

Starting with 9 counters, if Player A removes three or four, Player B can remove four or three to reach 2. If Player A removes one, Player B can also remove one to reach 7 which we already know is a winning positions.

(iii) Player B has a winning strategy, which is to ensure that, at the end of each of their turns, the number of counters is either a multiple of 7 or two more than a multiple of 7.

They can carry out this strategy as follows. Whenever Player A starts on a multiple of 7, Player B can ensure that the number of counters decreases by 5 or 7 (by taking $4, 4$ and 3 counters after Player A takes $1, 3$ and 4, respectively). Whenever Player A starts on a number which is 2 more than a multiple of 7, Player B can ensure that the number of counters decreases by 2 or 7 (by taking $1, 4$ and 3 counters after Player A takes $1, 3$ and 4, respectively).

Furthermore, Player A can never end numbers which are multiples of 7 or two more than a multiple of 7 (if they start on one of those numbers). This is because going between those numbers requires decreasing the number of counters by 2, 5 or 7, which cannot be done in one move.

In this game, Player A starts with 700 counters, which is a multiple of 7. Since the number of counters decreases every turn, it will reach 0 or 2 at the end of one of Player B's turns, and we know those are both winning positions.

5. A cuboid has side lengths $a - x$, a and $a + x$, where a, x are positive integers with $a > x$. The main body diagonal of the cuboid (connecting one vertex to the opposite vertex, passing through the centre of the cuboid) has length d.

This question asks you to prove that it is impossible for d to be an integer.

(a) Write down and simplify an equation relating a, x and d. [1 mark]

(b) Find all possible remainders that a square number can give when divided by 8. Hence prove that it is not possible for d to be an odd integer. [3 marks]

(c) Now suppose that d is an even integer.

(i) Prove that a and x must also be even. [1 mark]

(ii) Let $a = 2a_1$, $x = 2x_1$ and $d = 2d_1$. Find and simplify an equation relating a_1, x_1 and d_1. [1 mark]

(d) Complete the proof that it is impossible for d to be an integer. [4 marks]

COMMENTARY

This is a challenging problem, but the question structure should give you some hints for how to approach it. You will show that d can be neither even nor odd, but the two proofs are very different.

To prove that d cannot be odd, you need to look at the remainders when various square numbers are divided by 8. It turns out that there is only a small number of possible remainders, and none of the fit the equation found in part (a).

The proof that d cannot be even uses the technique called "infinite descent". This basically involves showing that, if there were an even integer d that satisfies the equation, we would able to keep dividing it by 2 indefinitely to get a new solution.

(a) Using Pythagoras gives

$$(a - x)^2 + a^2 + (a + x)^2 = d^2$$

which simplifies to

$$3a^2 + 2x^2 = d^2.$$

(b) Every number can be written as $8k + r$ where $r = 0, 1, \ldots, 7$. Squaring those shows that the remainders when they are dividing by 8 are $0, 1, 4, 1, 0, 1, 4, 1$. So the only possible remainders are 0, 1 and 4.

If d is odd, then d^2 gives remainder 1 when divided by 8. Since $2x^2$ is even, a must be odd and therefore $3a^2$ gives reminder 3 when divided by 8. The options for the remainder of $2x^2$ are $0, 2$ and 8 (which is same as 0).

Hence the possible remainders of $3a^2 + 2x^2$ are 3 and 5, so this can never equal d^2 and hence d can't be an odd integer.

(c) Now suppose that d is even.

(i) $3a^2 = d^2 - 2x^2$ and both terms on the right are even, so a must be even.

But then a^2 and d^2 are both multiples of 4 (as squares of even numbers), which means that $2x^2$ must also be a multiple of 4. This is only possible if x is also even.

(ii) Since all three numbers are even, we can write $a = 2a_1, x = 2x_1$ and $d = 2d_1$. Substituting into the equation gives

$$12a_1^2 + 8x_1^2 = 4d_1^2$$

which simplifies to

$$3a_1^2 + 2x_1^2 = d_1^2.$$

(d) The final equation obtained in (c) is the same as the original equation for a, x and d. This means that we could apply the same reasoning to conclude that a_1, x_1 and d_1 must all be even.

But then we can repeat the same reasoning indefinitely, dividing by 2 to get a new set of three integers every time. This is not possible because the numbers are getting smaller and so can't remain positive integers. The conclusion is that d cannot be even.

We already know that d cannot be odd. So it is not possible for d to be a positive integer.